Krishnaswami Alladi

Ramanujan's Place in the World of Mathematics

Essays Providing a Comparative Study

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Cover illustration: The eight great mathematicians encircling Ramanujan were chosen because of the special significance of their link with Ramanujan. The two at the bottom are Hardy (on the left) and Littlewood (on the right) of Cambridge University; they were the two who analyzed Ramanujan's letters, and came to the conclusion that he was a genius of first magnitude. Immediately above them are Rogers (on the left) and Schur (on the right) who had independently proved what are now called the Rogers-Ramanujan identities and went beyond these identities in their own ways. Above Rogers and Schur are Abel (on the left) and Galois (on the right), geniuses like Ramanujan who died very young, but had made path-breaking contributions in their youth. At the top are Euler and Jacobi, two of the greatest mathematicians in history, with whom Ramanujan was compared for sheer manipulative ability.

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Dedicated with affection to my parents (the late) Professor Alladi Ramakrishnan and Mrs. Lalitha Ramakrishnan, and my wife Mathura, for their constant encouragement and support

Foreword

This is an eminently readable, unique book. There are many books about Ramanujan: numerous biographies, studies of his mathematics and its influence, even a psychoanalytic study. This, however, is the first to provide what might be termed as a perspective overview.

This book is a collection of essays written by Krishna Alladi over more than two decades. Most of the essays originally appeared in *The Hindu,* one of India's leading newspapers. Each was written with the intent of providing a new way of thinking about Ramanujan.

Several of the articles are book reviews. Each of the books in question illuminates one aspect of Ramanujan, from his personal biography to his *Lost Notebook.* Included among the reviews is one devoted to the stage play, *Partition.* Each review gives a fair and often enthusiastic evaluation of the work in question. Alladi does note inaccuracies when they occur. For example, *Partition* suggests that Ramanujan was obsessed with Fermat's Last Theorem. As Alladi points out, this is far from the truth.

The *aforementioned correction* leads to another group of articles in the book which contrast Ramanujan with some other important mathematicians (including Fermat). These articles are really the heart of the book and are especially illuminating. They run from Galois (who died young like Ramanujan) to C. Jacobi (the pioneer of elliptic functions, a favorite topic of Ramanujan) to the little known L.J. Rogers (a Ramanujan contemporary of very similar talents). Alladi, who has worked in several areas of number theory and analysis, and who, as the editor of the *Ramanujan Journal*, is uniquely qualified to write these historical sketches which provide an unusual and compelling view of Ramanujan.

The remaining essays provide additional ways of looking at Ramanujan and his influence. These range from discussions of the prestigious SASTRA Ramanujan Prize to specific accounts of how current research has been based on foundations laid by Ramanujan.

This is a great book to "dip into." While the theme is clear and coherent, each article can be read separately. Each is designed for a wide readership and can be read with enjoyment and understanding by nonmathematicians. Each article will

give you a new look at Ramanujan. More generally you will come away with an enhanced appreciation of human greatness as expressed in the life and work of this Indian genius.

University Park, PA, USA

George E. Andrews
Member: National Academy of Sciences, USA
Past President: American Mathematical Society

Preface

Srinivasa Ramanujan is one of the greatest mathematicians in history. What makes Ramanujan unique is the spectacular and surprising aspect of almost every one of his discoveries and the manner in which he arrived at his results. Ramanujan's life story is awe inspiring, and he has the adulation the world over not just of professional mathematicians but of students and lay persons as well. Indeed, one of the best ways to attract bright young students to mathematics is to expose them to the enchanting world of Ramanujan's mathematics.

The Ramanujan Centennial in December 1987 was an occasion when mathematicians around the world gathered in India to pay homage to this singular genius. It was an appropriate time to discuss the significance of his contributions and to consider the ways in which his work would have impact on future research. The Ramanujan Centennial had a major influence on my academic life in ways more than one. I organized an international conference at Anna University in Madras during December 19–21, 1987, and spoke on my work on Probabilistic Number Theory, a subject whose origins can be traced to a fundamental paper that Ramanujan wrote with Hardy in 1917. But my attention was drawn to various aspects of Ramanujan's mathematics that directly emanate from his notebooks. In particular, in 1987, I decided to change the focus of my research from classical analytic number theory to the theory of partitions and q-hypergeometric series. I also decided that starting from the Ramanujan Centennial, I would write articles about Ramanujan of general interest and publish them regularly in newspapers and magazines in order to reach a wide audience. I used to lecture regularly about number theory in general and Ramanujan's mathematics in particular in high schools in Madras, but by writing these popular articles, I felt I could reach an even wider audience. This book contains the collection of all such articles I wrote since the Ramanujan Centennial in 1987.

In order to properly understand the significance of Ramanujan's work or to get an idea of the place he occupies in the world of mathematics, we need to compare his life and work with those of other mathematical luminaries whose life and contributions have things in common with Ramanujan. Thus after writing the opening article "Ramanujan—an estimation" in 1987 for the Centennial, and following it with an article "Ramanujan—the second century" in which I discuss briefly the im-

pact that various aspects of his research will have in the following decades, I wrote a series of articles on the lives and works of mathematical giants who had strong links with Ramanujan. This is the first group of articles in this book, and they all appeared in "The Hindu" India's National Newspaper, as birthday tributes to Ramanujan annually in December. I felt that these articles would not only be useful to inform the lay public about the many great mathematicians in history and the impact of their contributions, but that one could appreciate Ramanujan better by studying him in comparison with other eminent mathematicians. The Hindu sometimes modified the titles of my articles, as well as the text in some places. The articles are published here with their original titles I had and with my original text in full. I will now briefly describe why I chose to write about certain mathematicians.

G.H. Hardy, Ramanujan's mentor, has said that for sheer manipulative ability, Ramanujan can be compared with Euler and Jacobi. Also there is much in common regarding the kind of mathematics Euler, Jacobi, and Ramanujan worked on. Thus the separate articles on Euler and Jacobi not only describe their lives and achievements, but also compare their mathematical methods and results with those of Ramanujan.

In Hardy's own words, the British mathematician L.J. Rogers was not unlike Ramanujan both in talent and in the kind of infinite series identities he investigated. Indeed Rogers had proved (what is now termed as) the Rogers–Ramanujan identities about 15 years before Ramanujan discovered them. And Ramanujan's rediscovery of Roger's work was actually responsible for Rogers' belated recognition by the British mathematical community. Thus the opening article on mathematicians is the one on Rogers.

Major P.A. MacMahon, Hardy's mathematical assistant, studied the Rogers–Ramanujan identities combinatorially. He was the one who verified the celebrated Hardy–Ramanujan formula for the partition function. MacMahon was stationed briefly in Madras, India, while serving in the British army. In view of these strong links with Ramanujan, an article on MacMahon is included.

The theory of partitions can be broadly classified into four eras: (i) the era of Euler, the founder, (ii) the era of Sylvester, who improved significantly on the results of Euler by using combinatorial methods, (iii) the era of Ramanujan who transformed the subject gloriously, and (iv) the modern era. So I felt that an article on Sylvester would be most appropriate.

G.H. Hardy said that the real tragedy with Ramanujan's life was not his early death at the age of 32, but that during his most formative years, Ramanujan was sidetracked due to lack of formal training. Hardy noted that the best mathematics is done at a young age. He cited Evariste Galois and Neils Henryk Abel as mathematicians who died very young but had made monumental contributions. Thus in two separate articles, the lives and works of Abel and Galois are discussed. The article comparing Abel and Ramanujan was written shortly after the Abel Prize in mathematics was launched by the Norwegian Academy of Science and Letters.

The proof of Fermat's Last Theorem in 1994 created a sensation the world over since it settled a three hundred year old conjecture. I noted that there were some links between Ramanujan and Fermat, and so in early 1995, I wrote an article comparing Fermat and Ramanujan.

The passing away of Paul Erdös in 1996 marked the end of a great era. Erdös was the most prolific mathematician in history and influenced the academic lives of many including me. His path-breaking work on probabilistic number theory was inspired by the seminal 1917 paper of Hardy and Ramanujan. Thus the December 1996 article is a memorial tribute to both Erdös and Ramanujan.

No account of Ramanujan is complete without a discussion of his mentor G.H. Hardy, and so an article on Hardy is included in this series. Hardy's famous collaborator J.E. Littlewood studied Ramanujan's two letters of 1913 along with Hardy. It was Littlewood who said that every number is a personal friend of Ramanujan! Thus an article on Littlewood follows the article on Hardy.

The great German algebraist Issai Schur had independently proved the Rogers–Ramanujan identities and saw partition theoretic extensions of them. These identities are sometimes called the Rogers–Ramanujan–Schur identities. Thus my penultimate article on mathematicians is on Schur.

Robert Rankin, the Scottish mathematician, was one who knew Hardy, and lived in our time as well. He died just a few years ago. It was he who collected all the papers in G.N. Watson's office (after Watson's death) and sent them to the Wren Library in Trinity College, Cambridge, for preservation. Unknowingly, the last manuscript of Ramanujan (which Watson was studying) was in this collection, and so was placed in the Watson estate at the Wren Library and forgotten. This is what George Andrews unearthed as the Lost Notebook in 1976. Thus the final article on mathematicians is on Rankin.

Naturally, from time to time, in addition to writing about mathematicians, I wrote about certain aspects of Ramanujan's mathematics. These also appeared in The Hindu and form the second group of articles in this book.

Ramanujan's spectacular series for π are among those used in present day computer calculations of the digits of π; the first article in the second group is on the history of π and on Ramanujan's work on this fundamental mathematical constant.

Some of Ramanujan's most significant contributions are in the theory of partitions, an area he gloriously transformed with his magic touch. The article "Ramanujan and partitions" discusses some of his most startling results and their far reaching impact.

In 2005, Manjul Bhargava and Jonathan Hanke solved a fundamental problem on universal quadratic forms that was raised by Ramanujan. Bhargava presented this solution as the Ramanujan Commemoration Lecture at a conference in SASTRA University in 2005 that I helped organize, the conference at which he and Kannan Soundararajan were awarded the First SASTRA Ramanujan Prizes. So I sent a report to The Hindu about this conference and on the solution of this problem of Ramanujan. This is the final article in the second group in this book.

Periodically I was asked to review various books on Ramanujan. The first was an invitation to review "The Lost Notebook and Unpublished Papers" of Ramanujan published by Narosa that was released on Ramanujan's 100th birthday, December 22, 1987, by India's Prime Minister Rajiv Gandhi and handed over to Professor George Andrews, who unearthed Ramanujan's Lost Notebook at the Wren Library in Cambridge and who wrote a charming introduction to the Narosa publication;

my review of this book appeared in The Hindu in January 1988. Next was a review of Robert Kanigel's famous biography of Ramanujan entitled "The man who knew infinity"; my review appeared in The American Scientist in 1992. Finally I was invited by the American Mathematical Monthly to review the two prize winning books by Bruce Berndt and Robert Rankin entitled "Ramanujan—letters and commentary" and "Ramanujan—essays and surveys". All these reviews are the in third group of articles in this book. In this third group, I have included a short review of "Partitions—a play on Ramanujan" that I wrote for The Hindu in May 2003; I had the pleasure of seeing this play along with George Andrews at the Aurora Theatre in Berkeley. I was so impressed with the play, that I immediately sent a review to The Hindu.

Finally, during the Ramanujan Centennial, I decided that I should create something which will not only be a permanent memorial to Ramanujan but also continue to develop his mathematical contributions in the context of current research. Thus I decided to launch The Ramanujan Journal that is devoted to all areas of mathematics influenced by Ramanujan. My proposal to launch this journal received support from eminent mathematicians worldwide many of whom either served, or are now serving, on the editorial board. The actual process of launching a journal takes time. The first issue of this journal came out in 1997. This journal, which was originally launched by Kluwer and now published by Springer, has established itself as one of the major journals. My service to the profession as editor of The Ramanujan Journal naturally led me to be invited by SASTRA University in Kumbakonam, Ramanujan's hometown, to organize their annual conferences relating to Ramanujan's mathematics starting in 2003. Two years later, at my suggestion, SASTRA University generously launched the SASTRA Ramanujan Prize, an annual prize of $10,000 given to very young mathematicians for outstanding contributions to areas influenced by Ramanujan. This prize is now recognized as one of the top mathematical prizes in the world. The final group of articles concern my efforts regarding the Ramanujan Journal, the SASTRA conferences, and the SASTRA Ramanujan Prize.

Now that we are celebrating Ramanujan's 125th birthday in December 2012, we are reflecting on the progress achieved since the Centennial. The time seems appropriate to assemble these articles to provide an idea of the place Ramanujan occupies in the world of mathematics.

Before writing these articles, I used to send drafts of each of them to the Trinity of Ramanujan's Mathematics, Professors George Andrews (Penn State University), Richard Askey (University of Wisconsin), and Bruce Berndt (University of Illinois, Urbana) for their comments. I benefited immensely from their criticism and suggestions. In particular, I have had long discussions with Professor George Andrews about the themes and contents of these articles. I am grateful that this towering figure in the world of Ramanujan's mathematics has written a foreword to this book.

I am most grateful to Mr. N. Ram, Editor-in-Chief of The Hindu, and Mr. N. Ravi, Editor of The Hindu, for generously allocating space in their leading newspaper and publishing these articles annually since 1987. I am thankful to The Hindu, The American Mathematical Monthly, The Focus Magazine, and the American Scientist for giving permission to reproduce these articles in this book. The reader has to note

that these articles appeared at different points in time and in different venues, and so there will be overlap when these are read collectively in this book. But then each article is self-contained, and can be read independently of the others.

When I decided to take to a research career in mathematics in 1975, it was my father Professor Alladi Ramakrishnan who told me that in addition to doing research, I should be active in communicating mathematics to school students and to the general public. Thus it was with his encouragement and advice that I wrote these articles. Had he been alive today, he would have been so happy to see this book published. Equally happy to see these articles collectively published are my mother Mrs. Lalitha Ramakrishnan and my wife Mathura, and I appreciate their constant support of all my efforts.

Finally I thank Elizabeth Loew, Thomas Hempfling and Shamim Ahmad of Springer for their interest in publishing this collection of articles for the 125th birth anniversary celebrations of Srinivasa Ramanujan, and Donatas Akmanavičius for help with the production of this book.

Gainesville, FL, USA Krishnaswami Alladi

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The actual journal and newspaper reference for each article is given as a footnote to the respective article. The articles published in this book are in the original form written by the author. What was published in the above newspapers and journals were either abridged or slightly modified versions of the original articles.

Contents

Chapter 1
Ramanujan: An Estimation

Ancient India has a rich mathematical tradition. The Hindus understood the role of zero within the algebraic framework of numbers. This led them to the decimal system which was transmitted to Europe by the Arabs. As in other civilisations, astronomy provided the Indians a motivation for mathematical exploration. In studying the duration of the eclipses, Aryabhatta, Bhaskara and Bhahmagupta systematically investigated a class of equations in number theory. In the post-Newtonian era, although great strides were made in Europe, for various reasons, nothing of scientific significance emanated from India. With the emergence of Ramanujan during the beginning of this century, this long period of hibernation came to an end, and Indian mathematical research was rejuvenated mainly in the realms of analysis and number theory.

Ramanujan is admired, and rightly so, for having achieved so much in such a short lifetime that was filled with the impediments of poverty. But examples abound of persons who did outstanding work under the most formidable circumstances. What makes Ramanujan unique is that in spite of the trammels of superstition and orthodox traditions that surrounded him, his untutored genius produced mathematics of the very highest quality.

Contempt expressed by peers strikes a more cruel blow than poverty. When Euclidean geometry was considered to be the only sensible geometry, a young Hungarian Bolyai ventured dangerously against established beliefs and sent his work on non-Euclidean geometry to Gauss, the unquestioned leader of mathematics during the early eighteenth century. After a prolonged silence, Gauss replied that he too had made similar attempts but had given them up because they were of no consequence. Bolyai succumed to the severity of the verdict and did not live to see the day when Einstein utilised non-Euclidean geometry in the theory of relativity. On the contrary, Ramanujan was lucky to be recognised by the British mathematician Hardy.

This article appeared in the center page of *The Hindu*, India's National Newspaper, on 19 December 1987, for the Ramanujan Centenary. The article was accompanied by a short note by Hindu Reporter M. Prakash which appeared as a boxed inset.

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Is sound knowledge of related fields necessary for research? A moot question indeed! A sophisticated mathematician looks at a problem not in isolation, but in terms of its relationship with other questions and then chooses the appropriate tools to solve it. Sometimes when such approaches have not made headway, a radically new idea from a relatively inexperienced researcher produces a breakthrough. Ramanujan was an extreme example of a mathematician whose contributions were of such high calibre that they belied his lack of formal training. But the penalty he paid was that much of his work turned out to be rediscovery. Whether Ramanujan would have reached greater heights had he been provided rigorous training is debatable. Hardy was of the opinion that such training would have made Ramanujan less of a genius. Instead of taking sides on this issue, we make note that the brilliance of Ramanujan combined with the sophistication of Hardy was the key to their successful collaboration.

Mathematics has grown so vast and intricate that it is unlikely that a Ramanujan-like phenomenon will ever surface again as it will be difficult to make fundamental contributions without a clear understanding of connections between different fields. In the contemporary scene, Paul Erdös is one leading mathematician who defies sophistication. Now past seventy, Erdos continues to be the most itinerant of scientists. His unconventional approach is marked with such distinction that an "Erdosian proof" is instantly recognisable. He is one of the principal architects of probabilistic number theory whose origins can be traced back to a joint paper that Ramanujan wrote with Hardy in 1917.

Ramanujan has to be measured only by the impact of his contributions. One may feel that the most important papers are those that solve longstanding problems. These are significant, but so also are those that trigger new questions and open up fresh avenues of thought. The great mathematician Bernhard Riemann, who worked mainly in analysis, wrote just one paper in number theory with the intention of proving the prime number theorem that was conjectured nearly a century earlier by Gauss. Although Riemann did not prove this theorem, he created a totally new approach and raised several questions that have kept mathematicians busy ever since. Even Ramanujan pursued some ideas akin to those in Riemann's paper. Certain questions raised in that paper have been solved, but one known as "The Riemann Hypothesis" remains unsolved to this day, and the very effort to find a solution has been rewarding.

Ramanujan has also raised several fundamental questions that have engaged mathematicians for decades. Some of his papers have created fruitful areas of research such as the one which led to probabilistic number theory. In another paper he raised a problem which Hardy later called "The Ramanujan Hypothesis", and this was settled only recently by Pierre Deligne in Paris. That the solution came after more than half a century is a measure of the depth of the problem. Deligne was honoured with the Fields Medal which is awarded once every four years at the International Congress of Mathematicians. There is no Nobel Prize for mathematics, but the Fields Medal is considered to be equivalent in prestige although not as lucrative.

Ramanujan lives today through the many questions he has raised. Deligne's achievement is a monumental inscription to his illustrious memory.

100 Percent Pure Talent Paul Erdös, a distinguished member of the Hungarian Academy of Sciences, told The Hindu in Gainesville, Florida, USA, that he was greatly inspired by Ramanujan's work. He said that Ramanujan's contribution to number theory in collaboration with Hardy formed the basis for his work which led to the creation of probabilistic number theory.

Paul Erdös said that when Hardy was asked what was his greatest contribution to mathematics, he unhesitatingly said "The discovery of Ramanujan". Hardy told him that Ramanujan went far beyond his theorems.

Hardy once gave an estimation of mathematicians on the basis of pure talent on a scale of 1–100. Professor Erdös said that Hardy gave Ramanujan 100, 80 to the famous mathematician David Hilbert, 30 to colleague Littlewood, and only 25 to himself. "Although Hardy was modest in giving himself only 25, the fact that he gave 100 to Ramanujan revealed the regard he had for Ramanujan's work."

A man who lives by numbers, Prof. Erdös noted that Ramanujan was so multifaceted that no mathematician could fully decipher or comprehend all of his creations. "I wish I had a chance to meet Ramanujan. Unfortunately he died when I was seven." Incidentally Ramanujan's Centenary also coincided with the 75th birthday of Professor Erdös. His admirers are planning to celebrate it in a big way.

Hardy was of the opinion that education would not have made Ramanujan a greater mathematician; it might have stifled his genius. However Professor Erdös expressed the view that education would have proved a great deal more for Ramanujan. "He would not have wasted so much time rediscovering the work of other mathematicians."

M. Prakash
Reporter, The Hindu

Chapter 2
Ramanujan: The Second Century

When European mathematicians first came to know of Ramanujan's spectacular results during the early part of this century, they perceived him as a singular genius who produced numerous beautiful but mysterious identities. To a mathematician a result is mysterious if he is not able to understand it in terms of well-known theorems or see it as part of a general theory. Lacking formal education, Ramanujan was in no position to motivate his results or supply rigorous proofs. Even Professor G.H. Hardy could not fully understand many of these identities on infinite series and products. Although Hardy compared Ramanujan to Euler and Jacobi for sheer manipulative ability, he expressed the opinion that Ramanujan's results lacked the simplicity of the very greatest works. But during the last half a century, many of Ramanujan's identities have been studied in detail and put in proper perspective with respect to contemporary theories. Hence his results do not appear now to be quite that mysterious, and in fact by the time his centenary was celebrated, it became clear that his work compared well with those of the very greatest mathematicians. But the study of Ramanujan's formulae is by no means over. As Professor Atle Selberg of The Institute for Advanced Study, Princeton, remarked during the Ramanujan Centenary, it will take many more decades, possibly even more than a century, to completely understand Ramanujan's contributions. Mathematicians know well that Selberg is not given to hyperbole, and so this is very high praise! I will now describe some features of Ramanujan's work which continue to excite researchers today and will engage them in the near future.

Mock Theta Functions Ramanujan's work on mock theta functions is considered to be one of his deepest contributions. These results were discovered by him just before he died, and he communicated them to Hardy in his last letter dated January 1920. A major portion of The Lost Notebook is devoted to mock theta functions. In his letter Ramanujan listed several mock theta functions of orders three, five, and

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This article appeared in *The Hindu*, India's national newspaper, on December 22, 1991, on Ramanujan's 104-th birth anniversary.

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seven. Hardy passed on to Professor G.N. Watson the task of analysing Ramanujan's mock theta identities. Watson wrote two papers on this topic, the first of which was his presidential address to The London Mathematical Society entitled "The Final Problem: An Account of the Mock Theta Functions." Watson explained the choice of the title as follows: "I doubt whether a more suitable title could be found for it than used by John H. Watson, M.D., for what he imagined to be his final memoir on Sherlock Holmes." Watson's first paper (1936) dealt with mock theta functions of third order, and the second (1937) with those of fifth order. Watson did not consider the seventh-order functions, but these were investigated by Selberg in 1938. In the last two decades Professor George Andrews has analysed and explained combinatorially many of Ramanujan's mock theta identities. In collaboration with his former student Frank Garvan, Andrews was led to conjecture that some of Ramanujan's mock theta identities were equivalent to certain results on partitions (a partition of a positive integer n is a representation of n as a sum of positive integers not exceeding n). These were called the "Mock Theta Conjectures." These conjectures were settled by Dean Hickerson in 1989, after the Ramanujan Centenary. Just this year Andrews and Hickerson have completed the study of eleven identities of Ramanujan on sixth-order mock theta functions in The Lost Notebook. Another recent advance is the work of Henri Cohen who explained certain mock theta identities in the context of Algebraic Number Theory.

In spite of these breakthroughs, several fundamental questions remain. For instance, no one knows what Ramanujan meant by the "order" of a mock theta function. Ramanujan divided his list of functions into those of third, fifth, and seventh orders. Known identities indicate that these are related to the numbers 3, 5, and 7, but a precise definition of order is yet to be given. So for now, the order of a mock theta function is a convenient label which may or may not have deeper significance. Ramanujan had defined mock theta functions to be those satisfying two conditions. But no one has rigorously shown yet that any of these mock theta functions actually satisfy the second of Ramanujan's conditions. Also, in dealing with mock theta functions, special techniques have been used based on the specific function being discussed. There are attempts to find a unified approach to deal with mock theta functions like the theory of modular forms that is used in the study of theta functions.

Ramanujan's Congruences Some of the most surprising observations by Ramanujan concern congruences or divisibility properties for the partition function. Hardy had asked MacMahon to prepare a table of first two hundred values of the partition function using a certain formula of Euler. As soon as Ramanujan saw this table, he pointed out three congruences involving the primes 5, 7, and 11. The first congruence states that the number of partitions of an integer of the form $5n + 4$ is divisible by 5. For example, there are 30 partitions of 9, and 30 is divisible by 5. Hardy was simply stunned, because partitions represent an additive process, and so he did not expect such divisibility properties. MacMahon had prepared the table, and Hardy had checked it, but neither observed such a relation! Ramanujan had the eye for such connections, and this is an example of the element of surprise that is

present throughout Ramanujan's work. Ramanujan generalised his congruences to the powers of 5, 7, and 11. Watson (1938) proved the congruences for the powers of 5 and (in a slightly modified form) for the powers of 7; the congruences involving the powers of 11 were established later by Atkin.

In 1944, Freeman Dyson, then a young student at Cambridge University, conjectured a combinatorial explanation for the congruences involving the primes 5 and 7 using the concept of "rank" for partitions. Dyson published his conjecture in "Eureka", a Cambridge student journal. The Dyson rank conjectures were proved in 1954 by A.O.L. Atkin and H.P.F. Swinnerton-Dyer using the theory of modular forms. Dyson had pointed out that the rank does not explain the third (and deeper) congruence involving the prime 11, but he conjectured the existence of a statistic, which he called the "crank", that would explain the third congruence combinatorially. But he had no idea of what the crank would be. Freeman Dyson has humorously remarked that this was the only instance in mathematics when an object had been named before it had been found! The crank sought by Dyson was found in 1987, one day after the Ramanujan centenary conference in Urbana, Illinois, by Andrews and Garvan. The solution was based on Garvan's Ph.D. thesis at Pennsylvania State University.

Whenever Ramanujan pointed out a relation, it was usually one of many that existed, and often the most striking among those. It has been shown that the coefficients in the expansions of various modular forms satisfy such congruences. During the last two years, Garvan, who is now at the University of Florida, has developed the idea of the crank to combinatorially prove and explain many such congruences. Also, Garvan, Kim and Stanton have applied ideas from Group Theory to explain deeper congruences. Thus the study of Ramanujan-type congruences will continue to be an active line of research in the future.

Rogers–Ramanujan Identities This pair of identities (discovered independently by Rogers in 1894 and Ramanujan around 1910) are considered among the most beautiful in mathematics. The combinatorial description of the first identity is that the number of partitions of an integer n into parts differing by at least two equals the number of partitions of n into parts which when divided by 5 leave remainder 1 or 4. The second identity has a similar description. The simplicity of the identities belies their depth. Several proofs have been given, but none can be considered simple or straightforward. In an attempt to understand these identities, a rich theory has developed, concerning, on the one hand, partitions whose parts satisfy gap conditions and, on the other, partitions whose parts satisfy congruence conditions. For a quarter century beginning around 1960, considerable work has been done in this direction, especially by Professor Basil Gordon of the University of California, Los Angeles, and by Professors George Andrews and David Bressoud of Pennsylvania State University.

Rogers actually found several elegant companions to the Rogers–Ramanujan identities. Fundamental discoveries always find applications eventually. In 1979, the Australian mathematical physicist Rodney Baxter showed that the Rogers–Ramanujan identities and these companions are the solutions to the Hard Hexagon

Model in Statistical Mechanics. For this work, Professor Baxter was awarded the Boltzman medal of the American Physical Society. In recent years, identities of the Rogers–Ramanujan type have found more applications to problems in mathematical physics.

Ramanujan considered the Rogers–Ramanujan identities as arising out of a continued fraction possessing a product representation. It was Ramanujan's insight to have realised the importance of this continued fraction in the theory of modular forms. This is only one of many continued fractions studied by Ramanujan, but perhaps the most appealing. Professor Bruce Berndt of The University of Illinois has analysed several continued fractions of Ramanujan. These continued fractions can be approached in various ways and offer a wide range of problems for exploration.

Products of the Ramanujan type established for this continued fraction are of interest in themselves. In 1980, Andrews and Bressoud showed that there was a pattern among the coefficients of certain Rogers–Ramanujan-type products that had value zero. Professor Gordon and I have recently extended these results to general Rogers–Ramanujan-type products, and there is scope for more work in this area.

Special Functions Ramanujan wrote down several beautiful formulae involving various special functions (like the Beta and Gamma functions). For the past two decades, Professor Richard Askey of the University of Wisconsin, with his students and co-workers, systematically studied q-analogues of various special functions. In the course of this study, many of Ramanujan's identities found in his original notebooks and in the Lost Notebook were extremely useful. Many of the q-analogues found by Askey and others are now finding important applications in Physics, through the idea of Quantum Groups.

The Notebooks When Bruce Berndt began editing the notebooks of Ramanujan, he envisaged publishing three volumes. Springer-Verlag has brought out three volumes, but the work is not over yet. Professor Berndt has almost completed work on the fourth volume, and there will be a fifth! This clearly demonstrates the depth and scope of Ramanujan's contributions. It is now possible to offer courses on Ramanujan's work since much of his work has been edited and books available on the subject. Thus a greater number of bright students will take to a study of Ramanujan's formulae in the decades to follow. In this connection, Robert Kanigel's recent book "The man who knew infinity" will open the eyes of the general public to the wonder that Ramanujan was.

The Lost Notebook It was George Andrews who discovered the Lost Notebook in 1976 at the Wren Library in Cambridge University. Since then, he has analysed hundreds of incredible identities contained in it and published several papers on them, most notably in the journal Advances in Mathematics. On 22 December 1987, Ramanujan's hundredth birthday, the printed version of the Lost Notebook was released. Professors Andrews and Berndt are planning an edited version of the Lost Notebook, much like Berndt's edited version of the original notebooks. This project will have great impact in the coming decades.

The Undying Magic In closing we emphasise certain features about Ramanujan's mathematics.

Ramanujan had the knack of spotting seemingly unexpected relations. Thus there is always an element of surprise for someone who studies his work. Quite often, a closer analysis reveals that there are many more such relations and that Ramanujan was pointing out only the most striking cases. So, one begins to suspect whether Ramanujan had a method to generate such relations. In an effort to find such methods, interesting theories emerge, sometimes leading to connections between different areas of mathematics. Some of the most intriguing connections recently found are between root systems of Lie algebras and the theory of q-series and modular forms. This is the work of Professors V.G. Kac, I.G. MacDonald, and D.H. Peterson. Such connections not only enrich the two areas but also offer several fruitful research projects.

Ramanujan's mathematics remains youthful even in the modern world of the computer. His modular equations were used by Canadians Jonathan and Peter Borwein to calculate π (the ratio of the circumference of a circle to its diameter) to several million decimal places. The Borweins showed that these modular equations produce efficient algorithms to obtain approximations to π and other numbers. More recently, Ramanujan's transformations for elliptic functions were used by David and Gregory Chudnovsky to produce very rapidly convergent algorithms to compute π; in fact the Chudnovskys have now calculated π to the order of about a billion digits!

Finally there is a lasting quality about Ramanujan's mathematics and about fundamental research in general. In the mid-eighteenth century, the British mathematician Stirling did calculations to produce the table of logarithms. With the advent of modern computers, such tables are among the least useful possessions of a library. But the mathematics that went into the construction of such tables never loses its lustre. Indeed, Stirling had developed methods for the acceleration of convergence of series, and this has been the basis for William Gosper's recent program to generate identities using computer algebra packages like MACSYMA. Motivated by Gosper's ideas, Ira Gessel and Dennis Stanton have used q-Lagrangian inversion to generate many identities of the Rogers–Ramanujan type.

Thus Ramanujan has left behind enough ideas to keep mathematicians busy well into the twenty-first century. Professor Dyson has remarked that we should be grateful to Ramanujan not only for discovering so much but also for providing others plenty to discover! What sets Ramanujan apart from the rest of the mathematical giants is that feeling of astonishment he creates with his stunningly beautiful identities. Ramanujan is like a gem with many faces. His identities can be studied from different viewpoints. Each face of this gem dazzles the beholder with its array of colours!

Chapter 3
L.J. Rogers: A Contemporary of Ramanujan

Rogers

Most of us associate the name of L.J. Rogers with the celebrated Rogers–Ramanujan identities and rightly so, because these identities which are unmatched in simplicity of form, elegance and depth, are representative of the best discoveries by both these mathematicians. Although Rogers had proved these identities in 1894 nearly twenty years before Ramanujan discovered them, his work was neglected even by his British peers. Indeed, it was only after Ramanujan's rediscovery of

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Rogers' paper in 1917 that Rogers received belated recognition leading eventually to his election as Fellow of The Royal Society (F.R.S.) in 1924. In spite of this, the mathematical world remained largely unaware of the true significance of Rogers' work. We owe primarily to George Andrews and Richard Askey our present understanding of the range of Rogers' contributions to the theory of q-series and special functions.

L.J. Rogers was a first rate mathematician and a man of many talents ranging from music to linguistics. In Hardy's own admission, Rogers was a mathematician whose talents in the manipulation of series were not unlike Ramanujan's. For sheer manipulative ability, Ramanujan had no rival, except for Euler and Jacobi of an earlier era. But if there was one mathematician in Ramanujan's time who came closest to the Indian genius in his mastery over infinite series and products, it was Rogers. In this article I will describe some of the mathematical contributions of Rogers, their significance and impact on current problems and how they relate to Ramanujan's work. I first describe the fascinating personality of this multi-talented man. This article would not have been possible without the help of Professor George Andrews of Pennsylvania State University, who provided me with several documents relating to Rogers including the Obituary Notices of the Royal Society of 1934.

Man of Many Talents Leonard James Rogers was born in Oxford, England, on 30 March, 1862. His father J.E. Thorold Rogers was a well-known Professor of Economics. In his childhood Rogers had a serious illness and although he recovered completely, he was not sent to school. J. Griffith, an Oxford mathematician, noticed that Rogers had superior mathematical ability and taught him in his boyhood. Rogers had a brilliant undergraduate career at Oxford University. In 1888 an independent Chair of Mathematics was created at Yorkshire College (now the University of Leeds), and Rogers was appointed Professor. He held that position with distinction until 1919 when ill-health forced him to retire prematurely. In 1921 he returned to Oxford, where he lived in retirement until his death on 12 September 1933 at the age of 71.

Rogers was tall, loose limbed and a rather gaunt figure. He was bespectacled and had a beard. He was careless of his appearance and said that his drab clothes were in keeping with his complexion.

Rogers was extra-ordinarily gifted and indeed a genius. His interests extended well beyond mathematics into music, languages, phonetics, skating and even rock-gardens! In this sense he was very different from Ramanujan who had few interests outside of his obsessive love of mathematics. Whatever Rogers studied, he not only acquired a full knowledge of it, but had enough mastery to have it at his fingertips. Combined with this was his whimsical wit and ironic humour, heightened by his ability to keep a serious countenance. With such combination of talent and humour, he was in constant demand at various social gatherings.

Of his varied talents, special mention must be made of his ability to play the piano, helped by his long and nimble fingers. He was a multi-linguist and could

speak French, German, Italian and Spanish fluently. Some have attributed this to his interest in phonetics, and the study of various dialects gave him the opportunity of exercising his wonderful ear to the differences of sound. His students enjoyed listening to his lectures, and to some the boredom of mathematical calculations was relieved by his sparkling sense of humour.

Although he was admired and respected by those around him for his many-sided brilliance, he complained that people were ignorant of his real interest, namely, mathematics. Even British mathematicians of his day paid little attention to his papers, and it was not until Ramanujan drew attention to Rogers' work in 1917 that Hardy realised the fundamental nature of Rogers' contributions. Rogers was then conferred the Fellowship of The Royal Society in 1924. In spite of this recognition, in an obituary published in Nature in 1933, it was said that apart from the Rogers–Ramanujan identities, he found little of mathematical value. The writer even expressed the opinion that had not Rogers wasted his time with his other interests but approached mathematics with a single minded purpose, he would have achieved a good deal more and therefore could have been considered a success in life. The research of Andrews in the realm of q-series and that of Askey on special functions have demonstrated that these opinions on Rogers were far from the truth. In fact, in his fundamental papers Rogers had anticipated the discoveries of many noted mathematicians.

The Rogers–Ramanujan Identities During 1893–1895, Rogers published three memoirs in The Proceedings of The London Mathematical Society on the expansion of certain infinite products. In these papers the Rogers–Ramanujan identities and several related results are proved. Ramanujan discovered these two identities in India between 1910 and 1913 and communicated them in letters to Hardy. Ramanujan did not have a proof of these identities and could not supply one when asked by Hardy. Neither Hardy nor his British colleagues had any idea how to approach these identities. The combinatorial version of the first identity is as follows: *The number of partitions of a positive integer into parts which differ by at least 2 is equal to the number of partitions of that integer into parts which when divided by 5 leave remainder 1 or 4.* The second identity has a similar description. This combinatorial description is due to MacMahon and Schur. Neither Rogers nor Ramanujan viewed these identities in terms of partitions.

In 1917, while going through some old issues of the Proceedings of The London Mathematical Society, Ramanujan came across Rogers' papers accidentally. Hardy said that Ramanujan expressed great appreciation for the work of Rogers. A correspondence between Rogers and Ramanujan followed, resulting in a considerable simplification of the proof. At about the same time, the German mathematician I. Schur, who was cut off from England by World War I, discovered these identities independently.

Rogers alluded to his re-emergence ironically, as was characteristic of him, in a letter to F.H. Jackson dated 13 February 1917: " It was with a certain amusement that

a theorem which I proved nearly 24 years ago should have remained in obscurity so long and recently brought into prominence as a conjecture. MacMahon wrote to me on the publication of his book regretting that he overlooked my work before it was too late. Since then, I have other ways of proving both identities in a more direct way ... ."

In the last few decades several proofs of the Rogers–Ramanujan identities have been given. A major advance was made by Basil Gordon (University of California, Los Angeles) in 1961, who produced an elegant generalisation. Spurred by this, Andrews made great progress and discussed a whole class of related identities. Yet, none of these proofs of the Rogers–Ramanujan identities are simple. In some sense, the simplest proof so far is the one due to David Bressoud (Pennsylvania State University) in 1983. Although the identities have a combinatorial interpretation, no simple combinatorial proof is available converting partitions of one type to another. A combinatorial proof was given by Garsia and Milne in 1981, but it runs to 50 printed pages!

The Hard Hexagon Model In 1979, the Australian mathematical physicist Rodney Baxter was working on a problem in Statistical Mechanics concerning the behaviour of liquid helium over a graphite plate. The Rogers–Ramanujan identities arose as one set of solutions to the model he considered. While struggling to understand these identities he found a proof. Baxter then worked out the full set of solutions and six other companion identities came up. He then contacted George Andrews, the leading authority on this subject, who pointed out that all these identities were contained in Rogers' papers of 1893–1895. A fruitful collaboration between Andrews and Baxter followed, and they obtained significant extensions of Baxter's original model. Thus Rogers' work found application to a problem in Physics nearly a century later. For this work, Baxter was awarded The Boltzman Medal of The American Physical Society. In a series of lectures given in Tempe, Arizona, in 1985, jointly sponsored by The American Mathematical Society and The National Science Foundation of U.S.A., Andrews discussed the eight identities of Rogers and their role in Baxter's Hard Hexagon Model.

Of these six companion identities of Rogers, two of them bear remarkable resemblance to the original Rogers–Ramanujan identities. Andrews and Askey asked for a combinatorial explanation of these two identities of Rogers. An elegant explanation was found by Bressoud in 1978, and these ideas led him to provide combinatorial insights into many partition problems.

The Mechanism Rogers had beaten Ramanujan in his own game by proving two identities that Ramanujan could not! What was the mechanism behind his proof that worked so well? There were several ideas. Rogers viewed these identities as coming out of a process of comparing coefficients in certain general expansions. He considered various combinations of sine and cosine, the trigonometric functions, and compared expansions obtained in two different ways. Such tricks were right up Ramanujan's alley but had escaped his attention. These ideas of Rogers have proved extremely fruitful and have led to the discovery of many partition identities.

W.N. Bailey pursued certain ideas of Rogers and invented a powerful technique now known as The Bailey Chain. But Bailey did not realise the true potential of this technique which has now become the standard process to generate identities of Rogers–Ramanujan type. In two papers which appeared in The Proceedings of The London Mathematical Society (1951–1952), Lucy Slater carried out Bailey's programme and listed over one-hundred such identities.

Anticipated Others The Rogers–Ramanujan identities were not the only instance when Rogers anticipated the work of other great mathematicians. In their famous book "Inequalities" published by Cambridge University Press in 1934, Hardy, Littlewood and Polya observed that Rogers had discovered Hölder's inequality in 1888 one year prior to Hölder. In 1936, Atle Selberg (now at The Institute for Advanced Study, Princeton) discovered certain identities related to the number 7 just as the Rogers–Ramanujan identities are related to the number 5. Freeman Dyson (also at Princeton now) noticed in 1943 that these identities were buried in Rogers' papers of 1895 and 1917.

In his third memoir of 1895, Rogers had considered certain extensions of the famous Hermite polynomials. In 1978, Richard Askey and Mourad Ismail made a systematic study of various results on Special Functions in this memoir and pointed out that Rogers had anticipated the work of many noted mathematicians in this important area.

Invariant Theory During 1886–1887, Rogers wrote four papers in The Theory of Invariants, a subject dealing with algebraic expressions which remain invariant under certain transformations. Such invariance is of great interest to physicists, especially to those working in Relativity. In MacMahon's words, "The theory of invariants sprang into existence under the strong hand of Cayley, but that it emerged finally into a complete work of art for the admiration of future generations of mathematicians, was largely owing to the flashes of inspiration with which Sylvester's intellect illuminated it." Rogers' papers on Invariant Theory published in 1886 formed the main topic of lectures given that year by Sylvester, who was Savillian Professor of Geometry at Oxford University. Invariant Theory was one area where Rogers' work drew immediate attention.

False Theta Functions As is now well known, the mock theta functions are among Ramanujan's deepest contributions. In his last letter to Hardy dated January 1920, Ramanujan said that he had discovered a new class of functions called the mock theta functions which arise more naturally than the false theta functions of Rogers. While it is true that Ramanujan's discoveries on mock theta functions are deeper, the false theta identities of Rogers have beautiful partition theoretic interpretation as was shown by Andrews in his 1979 New Zealand lectures.

Continuing Rediscovery Thus the rediscovery of the mathematical contributions of Rogers is continuing, and he is finally getting the recognition he deserved. Rogers published 35 papers spanning q-series, invariant theory and elliptic function theory.

These papers contain important ideas which have sown the seeds of many developments in the succeeding decades.

It would be fair to say that had not Ramanujan unearthed the Rogers papers of 1893–1895, Rogers would not have been elected Fellow of The Royal Society. India owed a great debt to Cambridge University and to Professor Hardy in particular, for encouraging Ramanujan. And this debt has been repaid with Ramanujan's rediscovery of the work of Rogers, thereby showing the British mathematical community that they had a mathematician among them whom they needed to recognise!

Chapter 4
P.A. MacMahon: Ramanujan's Distinguished Contemporary

MacMahon

Percy Alexander MacMahon was an unusual and noteworthy British mathematician, unusual because he had a distinguished career both in mathematics and in the military. Well known as Major MacMahon, he made fundamental contributions to combinatorics and the theory of algebraic forms and received several honours from The Royal Society and The London Mathematical Society. He is best remembered for

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his work on symmetric functions, permutations and partitions. MacMahon's investigations in the theory of partitions not only brought him into contact with Srinivasa Ramanujan but, had a direct effect on some of the brilliant contributions of the Indian genius. In this article I will first describe briefly MacMahon's life in the military owing to connections with India. Next I will discuss some of his major contributions to mathematics and finally emphasise those which have connections with Ramanujan. For biographical facts, I have benefitted from two obituaries of MacMahon, one by H.F. Baker which appeared in 1930 in the Proceedings of The London Mathematical Society, and another by H.H. Turner presented to the Royal Astronomical Society of which MacMahon was president. I have also referred to the Collected Works of MacMahon which have been edited by George Andrews. Finally, I have profited from several stimulating conversations with Professor George Andrews of The Pennsylvania State University and comments from Professor Richard Askey of the University of Wisconsin.

Military Career and India P.A. MacMahon was born in Malta on 26 September 1854 as the second son of Brigadier General P.W. MacMahon. He went to school in Cheltenham. Following in his father's footsteps, he entered the Royal Military Academy in Woolwich in 1871 as a gentleman cadet. He was swiftly made a Lieutenant and posted in the 5th Brigade at St. Thomas Mount in Madras. His stay in Madras was brief because he was transferred to the 8th Brigade in Lucknow soon after. He was again transferred to Meerut in the Punjab but there he stayed for three years. In 1877 he was posted in the Northwestern Frontier Force in Kohat in the Punjab. For his service in the military, he was promoted to the rank of Captain in 1881 and Major in 1889, the title by which he was known since then even in the mathematical world. In 1882 MacMahon returned to his alma mater, The Royal Military Academy, as an Instructor in Mathematics, a post that he held until 1888. He was then appointed as Professor of Physics at The Royal Artillery College in Woolwich, a position that he held with distinction until 1898. From 1906 to 1920 he was Deputy Warden of Standards under the Board of Trade.

Contributions to Mathematics Ever since his youth, MacMahon was absorbed by mathematics. But it was his contact in 1882 with Professor George Greenhill of The Royal Artillery College that changed his whole life for he was exposed to the theory of algebraic forms which at that time was undergoing a full flight of development under masters like Cayley and Sylvester. (Incidentally, Greenhill may have also had an effect on Ramanujan, because according to Hardy, Greenhill's book on elliptic functions was perhaps the source in India from which Ramanujan acquired a knowledge of that wonderful subject). To quote Sir Joseph Larmor, "MacMahon threw himself with indomitable zeal and insight into the great problems of this rising edifice of science. In a short time he was being counted as conspicuous among the leaders largely by the invention of new methods." MacMahon's great strength was his mastery over combinatorial techniques. In particular MacMahon made fundamental contributions to the theory of permutations and the theory of symmetric functions with which it was intimately connected.

In simple terms, a *permutation* of a set is a rearrangement of its elements, and *symmetric functions* are those whose values do not change under permutations of the variables. Permutations and symmetric functions occur in a wide variety of settings in different areas of mathematics. In the theory of probability, which often deals with games of chance, one is always interested in the number of possible outcomes. Permutations often dominate such calculations. In linear algebra, while solving systems of equations, the *determinant* is of paramount importance. As is well known, a system of n linear equations in n unknowns has a unique solution if and only if the determinant of the matrix of coefficients is non-zero. And permutations play a crucial role in the definition of the determinant of a square matrix. As is known to most students of mathematics, symmetric functions arise while solving polynomial equations. More precisely, the coefficients of the polynomial can be expressed in terms of various symmetric functions involving the roots of the equation. In a paper that appeared in the Proceedings of The London Mathematical Society in 1884, MacMahon obtained an elegant extension of a famous formula of Newton involving symmetric functions and the powers of the roots of a polynomial equation.

MacMahon's influence can be seen throughout the theory of permutations. He studied special types of permutations such as those called *derangements*. These are permutations in which every object is in the wrong position. For example, consider n persons going to a restaurant and each turning in his hat at the entrance. When the guests are ready to leave, the clerk returns the hats at random. The question now is in how many ways can the hats be returned such that no guest receives his hat. What is asked here is the number of derangements of n objects. In a paper of 1912 that appeared in the Transactions of The Cambridge Philosophical Society, MacMahon obtained several pretty formulas involving derangements.

MacMahon realised that it was important to study indices of permutations. Roughly speaking an index of a permutation is a measure of how much things have gone wrong in the rearrangement. In a massive paper that appeared in 1913 in The American Journal of Mathematics, MacMahon studied a variety of indices: the greater index, the lesser index, etc. For example, in the permutation 2, 1, 3, 6, 5, 4 of the integers 1, 2, 3, 4, 5, 6, consider those pairs of consecutive terms (k, l) in the permutation for which k is greater than l and k comes before l. In the above example, the required pairs are $(2, 1)$, $(6, 5)$ and $(5, 4)$. For each such pair, note the position where the first entry k occurs and add up the values of these positions. That gives the *greater index* of the permutation. In the above example, the greater index is $1 + 4 + 5 = 10$. MacMahon established a variety of important results on the indices of permutations. In particular, he showed that the generating function of the greater index is the same as the generating function of the number of inversions, both being equal to the *q-multinomial coefficient*. MacMahon was the first to study systematically the combinatorial properties of the q-multinomial coefficients and in particular the *q-binomial coefficient*. Today we understand the q-binomial coefficient as arising out of the expansion of $(x + y)^n$, where the variables x and y satisfy the q-commutation relation $yx = qxy$. Such variations in the familiar commutative law of multiplication are of great interest to physicists, and conditions such as the q-commutation relation are the type one encounters in the study of *Quantum Groups* which play an important role in modern physics.

The results on derangements as well as other results on permutations were all special cases of a rather general result that MacMahon called as the *Master Theorem*, which he proved in his paper of 1912 in The Philosophical Transactions of The Royal Society of London. The master theorem asserts the equality of the coefficients of monomials in the expansion of certain products to the coefficients that occur in the expansion of the reciprocal of a certain determinant. MacMahon himself demonstrated the usefulness of this in two memoirs. Subsequently several proofs of the master theorem have been given the most important in recent years being that of Dominique Foata of the University of Strasbourg in France, who explained its real combinatorial significance.

MacMahon's results on symmetric functions were extensive, and the great majority of his papers relate to them in some way. The basic tool that he used in studying symmetric functions were the Hammond operators. One of the active areas of research today is algebraic combinatorics, and two of the most prominent mathematicians in this field currently are Professors Gian-Carlo Rota and Richard Stanley of the Massachusetts Institute of Technology. The widespread influence of MacMahon's research can be seen in algebraic combinatorics while studying group representations or the Young tableaux.

In 1915 and 1916, Cambridge University Press published his book on *Combinatory Analysis* in two volumes. These volumes contain an extensive treatment of symmetric functions, partitions, compositions, and his work on the Master Theorem and on the enumeration of multipartite numbers.

As professor of physics at the Artillery College in the 1890s, it is not surprising that MacMahon was interested in problems in physics as well. In particular MacMahon published a paper in 1909 in the London Astronomical Society Monthly Notices dealing with the determination of the apparent diameter of a fixed star. This was his only paper in Astronomy. The great British cosmologist Sir Arthur Eddington criticised MacMahon's technique, and this held back its application for quite a long time. We know now that MacMahon's technique was a good one and that he had been correct even in the details. At this juncture it may be worth pointing out that Eddington was also very critical of Chandrasekhar's idea that massive stars would eventually contract under the influence of their own gravity. As it turned out, not only was Chandrasekhar correct, but his ideas became the starting point for the study of neutron stars and black holes.

For his significant contributions to mathematics, MacMahon received several honours. He was conferred Fellowship of The Royal Society (FRS) in 1890. He was elected as President of The London Mathematical Society for a two-year term in 1894. He served as President of The Royal Astronomical Society in 1917–1918. In 1900, The Royal Society first gave him the Royal Medal and later in 1919, the Sylvester Medal. In addition he received the de Morgan Medal of The London Mathematical Society in 1923. In his retiring Presidential address to The London Mathematical Society he gave a superb review of the progress in Combinatory Analysis during the second half of the nineteenth century. He died in Bognor on Christmas day in 1929.

Connection with Ramanujan It was MacMahon's work in the theory of partitions that brought him into contact with Ramanujan. A *partition* of an integer is a representation of that integer as a sum of positive integers, two partitions being considered the same if they differ only in the order of the parts. Thus we do not distinguish between $2+1+1$ and $1+1+2$ as partitions of 4. So the five partitions of 4 are 4, $3+1$, $2+2$, $2+1+1$ and $1+1+1+1$. If we make a distinction between two representations having the same parts but occurring in different order, then we are dealing with *compositions*. MacMahon studied both partitions and compositions combinatorially. Whereas MacMahon used combinatorial techniques, Ramanujan employed analytic methods. MacMahon was very quick in computation and was equal to Ramanujan in this regard. The number of partitions of an integer n, denoted by $p(n)$, grows quite rapidly, and MacMahon used a recurrence formula of Euler involving the pentagonal numbers to compute the first two hundred values of the partition function. This computation of MacMahon had a direct effect on Ramanujan's work as we shall see presently.

The Hardy–Ramanujan Formula The asymptotic formula obtained by Hardy and Ramanujan for the number of partitions of an integer is the finest example of what was achieved when the brilliance of Ramanujan combined with the sophistication of Hardy. Ramanujan realised even in India prior to his departure to England that there are analytic series representations for $p(n)$ and other related functions. It really stunned Hardy when Ramanujan asserted that such a series would give the exact value of $p(n)$. Hardy initially disbelieved this claim because $p(n)$ being a function on the integers has jumps, whereas the series that Ramanujan wrote down involved continuous functions. Ramanujan's spectacular discoveries are full of unexpected relations such as this. But Ramanujan had no proof of this claim. What was required was an ingenious use of the theory of functions of a complex variable, and here the mastery of Hardy over such sophisticated techniques was crucial. Hardy and Ramanujan obtained a series representation for the number of partitions of an integer n by an entirely new technique called the *circle method*. This method was later improved substantially by Hardy and his distinguished colleague Littlewood and is the standard technique used now in a wide variety of additive problems in number theory. Hardy and Ramanujan wanted to check their formula numerically, and for this, MacMahon's table of values for $p(n)$ proved useful. The value $p(200) = 3972999029388$ given by the series coincided with the value that MacMahon had calculated using Euler's recurrence. Actually a significant improvement over the Hardy–Ramanujan formula for $p(n)$ was obtained subsequently by Hans Rademacher, but I will discuss this in a separate article devoted to the life and contributions of Rademacher and highlight connections with Ramanujan's work.

The Ramanujan Congruences MacMahon's table of values of the partition function had another effect on Ramanujan's work, this one being much more dramatic. As soon as Ramanujan saw MacMahon's table, he wrote down three *congruences* (divisibility properties) for the partition function. The first of these congruences asserts that the number of partitions of an integer of the form $5k+4$ is a multiple of 5. For example, $p(4) = 5$ and $p(9) = 30$ are both multiples of 5.

When MacMahon prepared the table, he wrote down the values of $p(n)$ in columns of five entries in each column. Thus the first column contained entries from $p(0)$ up to $p(4) = 5$. The second column contained entries from $p(5) = 7$ up to $p(9) = 30$, and so on. Thus the last entry in each column was the value of the partition function at an integer of the form $5k+4$. So the last entry of each column was a multiple of 5. Ramanujan instantly noticed this and wrote down his first congruence. So this raises the question why MacMahon who had prepared the table, and Hardy who had verified the correctness of the entries, did not discover this congruence. The answer lies in the fact that neither MacMahon nor Hardy had the slightest suspicion that such remarkable results might hold. Partitions after all represent additive processes, and so why would anyone (but Ramanujan!) expect divisibility properties for partitions? Ramanujan had the eye for the unexpected and therefore discovered the congruence. This single incident is enough to demonstrate that Ramanujan's mind worked so differently from those of his peers. Ramanujan went on to write down two more congruences, the second involving the prime number 7 and the third involving the prime number 11. In fact Ramanujan stated a general formula for congruences involving powers of 5, 7 and 11. The study of Ramanujan-type congruences for various partition functions and related objects is an active field of research today. And all this started with Ramanujan seeing MacMahon's table of values of the partition function.

The Rogers–Ramanujan Identities In the entire theory of partitions and q-series, the two Rogers–Ramanujan identities are unmatched in simplicity, elegance and depth. Ramanujan discovered these results in India sometime around 1910, prior to his departure to England. He communicated them in a letter to Hardy in 1913, but he could not prove them. The identities interested Hardy and other British mathematicians immensely. In particular, MacMahon immediately recognised their combinatorial significance. The combinatorial interpretation of the first identity is that the number of partitions of an integer into parts differing by at least two equals the number of partitions of that integer into parts of the form $5k+1$ or $5k+4$. The second identity has a similar interpretation. But MacMahon could not prove the identities, and so he stated them as unsolved problems of Ramanujan in his treatise on Combinatory Analysis. Later in 1917, while going through some old issues of the Proceedings of The London Mathematical Society, Ramanujan came across some papers of the British mathematician L.J. Rogers in which these and related identities were proved. From then on they were known as the Rogers–Ramanujan identities. But neither Rogers nor Ramanujan realised their combinatorial significance. Subsequently, it was found that the German mathematician I. Schur had proved the identities in 1917 and given their combinatorial interpretation. There was little communication between mathematicians in England and Germany at that time due to World War I. But Schur's proof was also analytic. It should be emphasised that even today there is no simple combinatorial proof of the Rogers–Ramanujan identities. Nevertheless, with MacMahon and Schur emphasising their combinatorial significance, a new avenue of exploration in the theory of partitions opened up, one that is active even today.

A Parity Question In connection with the divisibility properties of the partition function, one is most naturally led to the question as to when $p(n)$ is odd and when it is even. This apparently simple problem is very deep and is unsolved even today. This question interested Ramanujan who asked MacMahon if he knew anything about it. In a paper published in 1921 in The Proceedings of The Cambridge Philosophical Society shortly after Ramanujan's death, MacMahon gave a procedure to quickly determine the parity of $p(n)$ for any given numerical value of n and illustrated this for $p(1000)$. But he did not prove any general results on this parity problem. It is known that infinitely many values of the partition function are odd and infinitely many are even. But it is not known whether roughly half the values are odd and the other half are even although this is conjectured to be the case.

MacMahon's Stature It is well accepted that MacMahon's premier contribution was the theory of perpetuants and the theory of symmetric functions that came out of it. Symmetric functions appear today in a variety of forms in different fields such as in group representations and K-theory. Speaking of the contributions of MacMahon, the famous mathematician Gian-Carlo Rota of The Massachusetts Institute of Technology points out that often in our desire for abstraction and modernisation we lose track of the wonderful classical ideas from which such results sprung forth. He says: "In mathematics it would be hard to find a more blatant instance of the regrettable state of affairs than in theory of symmetric functions... each generation rediscovers them and presents them in the latest jargon... Today it is K-theory, yesterday it was categories and functors, and the day before, group representations. Behind these and other attractive modern theories stands one immutable source: the ordinary, crude definition of the symmetric functions and the identities they satisfy."

In 1978, while editing the Collected Works of MacMahon, George Andrews observes: "MacMahon's researches in Combinatorics were ahead of his time. The significance of MacMahon's work was not clear to the next generation of mathematicians... Within the last twenty years, Combinatorics has undergone a remarkable renaissance and a random check through the Science Citation Index shows that MacMahon is no longer neglected."

The ordinary view that pure mathematics was an isolated subject was thoroughly distasteful to MacMahon. He had an earnest faith that its relationship with other domains, though they might be obscured for the moment, would ultimately emerge in the clear light of day. His attitude is best expressed in his own words: "I do not believe that any branch of science or subject of scientific work is destitute of connection with other branches. If it appears to be so, it is especially marked out for investigation by the very unity of science." The manifold uses of symmetric functions in many areas of mathematics and other fields today are a wonderful testimony to his belief in the unity of science.

Chapter 5
Fermat and Ramanujan: A Comparison

Fermat

Although Pierre Fermat (1601–1665), one of the founding fathers of Number Theory, and Srinivasa Ramanujan (1887–1920), the legendary Indian genius, are separated by centuries, there are many similarities between the two in style and substance. An important part of the legacy of both Fermat and Ramanujan are their many observations recorded informally which have inspired several succeeding generations of mathematicians. Fermat's Last Theorem, which until recently was the most famous unsolved problem in mathematics, was just a marginal entry made

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by Fermat in a book written by Bachet on the work of the Greek mathematician Diophantus. The statement of the Last Theorem is that for any integer n greater than 2, there are no positive nth powers which are the sum of two other positive nth powers. Fermat claimed that he had a truly marvelous proof of this assertion but unfortunately the margin was too small to contain it! This naturally added a spirit of intrigue to the problem. The simplicity of Fermat's Last Theorem belies its depth and difficulty. For the last three centuries, mathematicians both amateur and professional, have attempted to find a proof of Fermat's assertion but did not succeed. These attempts however did yield plenty of new techniques which have proved immensely useful elsewhere. For example, the subject Algebraic Number Theory was born owing to efforts by Kummer and others to understand the unique factorisation property in very general settings, being motivated to study this question while attempting to solve Fermat's Last Theorem. The proof of Fermat's Last Theorem announced by Andrew Wiles in June 1993 and now completed by Wiles and Taylor is the culmination of years of effort by many illustrious mathematicians and is the result of the fusion of the Theory of Elliptic Curves and Number Theory. In a similar vein, Ramanujan's incomplete entries in his two notebooks and in the Lost Notebook have engaged mathematicians since the beginning of this century. Several branches of mathematics such as Number Theory, The Theory of Partitions, The Theory of Modular Forms, The Theory of Elliptic and Theta Functions, Hyper-Geometric Series and others have been enriched out of attempts to understand Ramanujan's jottings.

Both Fermat and Ramanujan communicated their wonderful findings in letters. Ramanujan wrote letters to mathematicians in England, desperately seeking recognition for his work. Fortunately, Hardy responded favourably. Fermat communicated regularly with his French peers Pascal and Mersenne among others, as well as with British mathematicians. Fermat's challenge to the British mathematicians was what ultimately led to the complete solution of what is now known as Pell's equation. The name Pell's equation is due to Euler who was under the mistaken impression that the British mathematician Pell had done most of the work on this; but we know now that several centuries earlier, the Indian mathematicians Bhaskara and Brahmagupta had made significant progress on such questions.

There are many number theoretic problems which interested both Fermat and Ramanujan. Fermat stated that every positive integer is a sum of no more than three triangular numbers, four squares, five pentagonal numbers, and so on. Special types of numbers like these had been of interest since the days of the Pythagoreans, but no one before Fermat had made such a fundamental observation about them. This assertion of Fermat attracted the attention of many outstanding mathematicians. Gauss gave a proof of the statement that every positive integer is a sum of no more than three triangular numbers, while Lagrange proved the assertion about sums of four squares. The general assertion that every positive integer is a sum of n or fewer n-gonal numbers was established by Cauchy.

Ramanujan was also interested in the representation of integers as sums of squares. But he viewed it from an entirely different angle, in terms of infinite series identities for theta functions. Although Ramanujan seldom stated number-theoretic forms of his identities, such interpretations were natural consequences of his results.

Just as Carr's Synopsis, the first book to make an impression on Ramanujan, so strongly influenced his style of writing, Bachet's Diophantus was the book that dominated Fermat's mathematical life. And it was in this book that Fermat copiously made marginal notes, commenting on possible extensions and improvements of results contained therein. In particular, Pythagorean triangles fascinated Fermat.

It is a fact well known to all high school students that if x, y and z denote the lengths of the three sides of a right-angled triangle with z being the length of the hypotenuse, then $x^2 + y^2 = z^2$. Pythagorean triangles are those where the x, y and z are integers without a common factor. An example is the triangle with sides 3, 4, and 5. Another example is provided by the triple 5, 12 and 13. There are infinitely many Pythagorean triangles, and a formula to generate all of them has been known since antiquity and can be found in Bachet's book. Fermat was interested in finding whether there were any Pythagorean triangles whose area was also a square, and this problem was not discussed by Diophantus or Bachet. Fermat proved by the *method of infinite descent* that such triangles could not exist. The key idea in the method of infinite descent is to show that any positive solution will generate a smaller such solution. By iteration, an infinite sequence of decreasing positive integer solutions would be generated, and this is clearly impossible.

Fermat was not the first person to investigate areas of Pythagorean triangles. As early as the tenth century A.D., the Arabs were interested in determining those numbers which arise as areas of Pythagorean triangles. Fermat proved that squares cannot be areas of Pythagorean triangles. This apparently idle question as to which rational numbers can be realised as areas of right-angled triangles with rational sides has been shown to have significant implications in the modern theory of elliptic curves.

With the exception of The Last Theorem, Fermat is most famous for his method of infinite descent. Fermat used the method of infinite descent to show that various equations do not have solutions among the positive integers, and perhaps believed that this method could be used to give a "truly marvelous proof" of the Last Theorem. In the course of proving that it is impossible to have Pythagorean triangles whose area is a square, Fermat showed that $x^4 + y^4 = z^4$ has no positive integer solutions. He then observed that he could also apply the method of descent on the equation $x^3 + y^3 = z^3$, which is the first case of The Last Theorem. But it was left to Euler to supply the arguments for this case.

In contrast, Ramanujan was interested mainly in equations which had solutions, and especially providing algorithms or formulas for the solutions. The famous Ramanujan taxi-cab number 1729 is a solution to what appears like a mild variation of the Fermat equation. Indeed 1729 is interesting because $1729 = 12^3 + 1^3 = 10^3 + 9^3$, and this is a solution to $x^3 + y^3 = z^3 + w^3$. In other words, addition of an extra variable w to the Fermat equation provides a solution. Euler had provided a formula for the general solution to this equation, but in Ramanujan's third notebook there is also a formula for the general solution to the taxi-cab equation which is in some ways more elegant than Euler's.

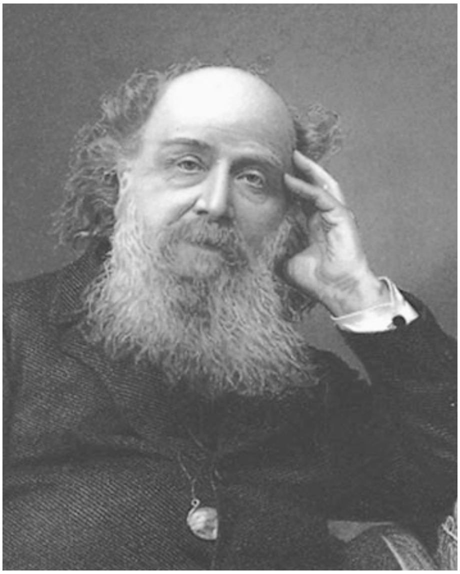
During the Ramanujan Centennial, Atle Selberg of the Institute for Advanced Study at Princeton pointed out that raising fundamental questions is just as important

as solving long-standing problems. Both Fermat and Ramanujan have raised several important questions which have engaged mathematicians of the highest calibre in the decades that followed. Fermat's assertions interested Euler, who systematically supplied proofs for many of them. And in doing so, Euler noticed improvements and generalisations. Gauss then took over where Euler had left off. Number Theory as it is taught today, is based on Gauss' *Disquisitiones Arithmeticae*, which is the finished product of the foundation laid by Fermat and the structure erected by Euler. Ramanujan's observations have been food for thought since the beginning of this century. At first, Hardy, Watson and Mordell supplied proofs and explanations for many of Ramanujan's observations. In the past few decades, the work of Erdös, Selberg, Deligne, Askey, Andrews, Berndt and others have revealed the grandeur of Ramanujan's discoveries. However, a complete understanding of Ramanujan's writings will take many decades, possibly more than a century.

The importance of the work that has been generated by a study of the writings of Fermat and Ramanujan can be judged by the kind of recognition that the international mathematical community has given to such efforts. In 1986, Gerd Faltings of Germany was awarded the Fields Medal (the equivalent of the Nobel Prize in mathematics for prestige) for proving the Mordell–Weil conjecture. From this work of Faltings it followed that every Fermat equation had at most a finite number of solutions. Faltings' methods have now led to the creation of a new field known as Arithmetic Geometry. Similarly, Andrew Wiles' proof of Fermat's Last Theorem contains many new ideas which will be developed to yield more results in the future. The most famous of Ramanujan's problems was what Hardy called *The Ramanujan Hypothesis* concerning the size of Ramanujan's tau function. Pierre Deligne was honoured with the Fields Medal in 1978 for proving the Ramanujan hypothesis.

Freeman Dyson of The Institute for Advanced Study in Princeton has said that we should be thankful to Ramanujan for not only discovering so much, but also leaving plenty for others to discover! Similarly, we should be grateful to Fermat for raising questions which have kept mathematicians busy for the past three centuries.

Chapter 6
J.J. Sylvester: Ramanujan's Illustrious Predecessor



Sylvester

The Theory of Partitions, an active area of research today, has gone through four major periods of development. The first era is that of Euler who founded the subject in the 18th century. The second period in the latter half of the 19th century is that of Sylvester who extended the work of Euler considerably by employing combinatorial techniques. Then Ramanujan entered the scene at the beginning of this

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century, and he totally transformed the subject by discovering several surprising results including an asymptotic formula (along with Hardy) and beautiful congruences (divisibility properties). Finally, we are witnessing the modern era when the theory of partitions is being linked to many branches of mathematics, computer science and even to physics. Although the theory of partitions is intertwined with the theory of q-series, this classification into four eras is for partitions only and does not include the periods of development due to Heine and others in the theory of q-series. In this article I shall describe the life of Sylvester, his major contributions to partitions, and discuss connections with certain aspects of Ramanujan's work. I have obtained biographical details from MacMahon's obituary of Sylvester which appeared in Nature (1897) and in The Proceedings of The Royal Society (1898). For a description of Sylvester's work on partitions, I have used a paper of George Andrews which appeared in a volume commemorating the centenary of The American Mathematical Society. Finally, for illuminating connections between Sylvester's work and some identities of Ramanujan, I have benefited from Andrews' New Zealand Lectures (1979) and several stimulating conversations with him.

Foundation Laid by Euler Leonard Euler (1703–1783), the most prolific mathematician in history, was the founder of the theory of partitions. Euler noticed that beautiful results on partitions could be proved by using what are called generating functions. These generating functions due to Euler are among the most fundamental examples of q-series. A partition of an integer is a decomposition of it into other positive integers, two partitions being considered the same if they differ only in the order of their parts. For example, $4, 3 + 1, 2 + 2, 2 + 1 + 1, 1 + 1 + 1 + 1$ are the five possible ways to partition 4. One of Euler's beautiful discoveries is that the number of partitions of an integer into odd parts equals the number of partitions of that integer into non-repeating parts. We shall refer to this result as Euler's theorem. Although Euler gave partition interpretations to some of his identities, he used generating functions to prove them. He was a master in the manipulation of infinite series, rivalled only by Ramanujan. Sylvester's principal contribution to partitions was to improve substantially upon Euler by combinatorial methods, an approach which was aesthetically pleasing because partitions are combinatorial objects. Indeed, the combinatorial theory of partitions began with Sylvester and his students at Johns Hopkins University in the United States. But Sylvester moved to Johns Hopkins only at the age of 61. I shall therefore first describe Sylvester's life and contributions before Johns Hopkins.

Sylvester Before Johns Hopkins James Joseph Sylvester was born in London on 3 September 1814 as the youngest son of Abraham Joseph. He had four brothers and two sisters. His oldest brother, who assumed the name Sylvester, was successful in the United States, and all the younger brothers followed his example in adopting the same name. From the age of 6 to 12, James went to a Jewish boarding school run by Mr. Neumegen, who being a good mathematician, recognised the boy's talent and nurtured it. When Dr. Gregory of The Royal Military Academy in Woolwich tested Sylvester at the age of 11, he said that the boy was so talented that he deserved special attention. In 1827, Sylvester became a student in The Royal Institution in Liverpool. There he won several prizes in mathematics. But his stay in this school

was not happy because he was constantly teased by his schoolmates of his Jewish background. Sylvester fought several battles with his schoolmates in defense of his religion. Unable to tolerate this mistreatment any longer, he ran away to Dublin with very little money. There, by an unusual and pleasant stroke of fate, he met a relative who helped him and persuaded him to return to Liverpool.

Sylvester joined St. John's College in Cambridge in 1831 at the age of 17. There he came out first in his class during the first year, but became ill in the second year and so had to stay at home. Although he returned to college the very next year, he was plagued by illness and so did not complete his studies until 1837, but finished as the Second Wrangler.

Soon after this Sylvester presented a paper criticising Euclid's definition of a straight line as a length without breadth. From this and his other publications it was clear that a mathematician of first magnitude was on the rise. He was appointed in 1837 to the Chair of Natural Philosophy at The University College, London, with his friend de Morgan occupying the Chair of Mathematics. During his tenure at the University College, he published a series of papers in the Theory of Equations and Elimination. Sylvester today is most known for his work in the Theory of Invariants, and E.T. Bell in his delightful book called Men of Mathematics devotes an article on Sylvester and Cayley entitled *Invariant Twins*. Although Sylvester and Cayley were major figures in Invariant Theory and were friends, they had very different personalities and lifestyles. At one time Sylvester and Cayley were roommates in London when both were bachelors. Cayley then married and settled down in Cambridge. Sylvester never married, was never at one place for any great length of time, and had a more tumultuous life.

In 1841 Sylvester accepted a Professorship at The University of Virginia in America despite warnings from his friends who knew about his strong feelings against slavery. The first few years were not mathematically productive. But from 1844 onwards Sylvester wrote several fundamental papers on a variety of topics in Virginia, and this attracted mathematicians of esteem like Poncelet, Dirichlet, Kummer and others. Unfortunately his stay in Virginia came to an abrupt and unpleasant end when he got involved in a sparring incident with a student who was unhappy with the grade that Sylvester gave him. Actually the angry student prepared an ambush, and Sylvester's sparring was only a spontaneous reaction, but the incident was quite embarrassing and forced the professor to leave the United States. Fortunately several leading mathematicians in Europe were aware of Sylvester's stature, and the letters they wrote testifying to his eminence led to Sylvester's appointment to a Professorship at The Royal Military Academy in Woolwich in 1855.

The years in Woolwich were scientifically the best period for Sylvester. Seated under a walnut tree in his garden, Sylvester made some of his greatest discoveries. According to MacMahon, Sylvester's best paper was written at that time and it appeared in The Philosophical Transactions of The Royal Society in 1864. This paper dealt with complex roots of certain algebraic equations. This problem was first considered by Newton and had defeated even giants like Euler, but Sylvester was successful with it. Sylvester also worked on partitions during this period, prior to his stay at Johns Hopkins. MacMahon observes that in June/July 1859, Sylvester

delivered a series of seven lectures on partitions at Kings College, London. Printed outlines of these lectures were distributed to the members of the audience and some others. Much later, in 1897, The London Mathematical Society published these lecture outlines, and they attracted considerable attention.

Unfortunately, at Woolwich Sylvester was in constant dispute with the authorities who considered his manners eccentric and irritable. Therefore, in 1870, even though his research was still going strong, he was forcibly "super-annuated" at the age of 56. But in a few years a new direction in his career was about to begin.

Sylvester at Johns Hopkins In 1875, the Johns Hopkins University was founded in Baltimore, Maryland, under the brilliant leadership of President Gilman, who felt that the ideal way to start would be to appoint two outstanding professors, one in Classics and the other in Mathematics. This way they would be able to ensure excellence without a huge financial commitment. The exact words of Daniel Gilman were: "Enlist a great Mathematician and a distinguished Grecian; your problem will be solved. Such men can teach in a dwelling house as well as in a palace. Part of the apparatus they will bring, part we will furnish. Others will follow." So the famous Joseph Henry sent a letter to Sylvester in August 1875 inviting him to build the mathematics department at Johns Hopkins and offered him a handsome salary, $5000 per annum paid in gold. Sylvester was free to teach whatever he wanted in any manner he saw fit. It was a bold experiment in the educational method. Sylvester, who was still full of fire and enthusiasm, once again crossed the Atlantic.

At Johns Hopkins University, Sylvester founded The American Journal of Mathematics, one of the leading mathematical research journals today. He found the freedom given to him to be so conducive to research. Perhaps the happiest years of his life were the ones he spent at Johns Hopkins. He lectured on a wide variety of topics, the choices being made at the spur of the moment on what he was thinking at that time. Undoubtedly this was difficult for some, but the spontaneity had a profound positive effect on many of his students and associates. The experiment in education taken by the regents of the university was a success. The contents of many of Sylvester's lectures became papers in the newly formed journal. In fact Sylvester published as many as 30 long papers in this journal during the first five years.

Many students and associates of Sylvester in Baltimore have commented on his lecturing style. W.P. Durfee has written: "His manner of lecturing was highly rhetorical and elocutionary. When about to enunciate an important or remarkable statement, he would draw himself up till he stood on the very tip of his toes and in deep tones thunder out his sentences. He preached at us at such times, and not infrequently he wound up quoting a few lines of poetry to impress upon us the importance of what he had been declaring."

A.S. Hathaway says: "I can see him now with his white beard and a few locks of grey hair, his forehead wrinkled o'er with thoughts, writing rapidly his formulae on the board, sometimes explaining as he wrote, while we, his listeners, caught the reflected sounds from the board. But stop, something is not right; he pauses, his hand goes over his forehead to help his thought; he goes over the work again, emphasises the leading points and finally discovers his difficulty ... But at the next lecture, we

would hear of some new discovery that was the outcome of that difficulty and some article for the journal that he had begun."

It was during this period that Sylvester wrote a long and important paper entitled "A constructive theory of Partitions arranged in three Acts, an Interact and an Exodion" (American Journal of Mathematics, 1882). This paper contains several seminal ideas of Sylvester and members of his group of whom the most prominent were F. Franklin and W.P. Durfee. In this massive paper, Sylvester laid the foundations of the combinatorial theory of partitions by studying properties of partition graphs. He provided combinatorial proofs of Euler's assertions and extended them. In particular he obtained a significant refinement of Euler's theorem on odd parts and non-repeating parts. Contained in this paper is Franklin's famous proof of Euler's celebrated Pentagonal Numbers Theorem. This proof by Franklin is considered to be the first significant achievement in American mathematics. Yet another idea in this paper which is useful even today, is the concept of a Durfee square (named after W.P. Durfee).

By studying Durfee squares for partitions into non-repeating parts, Sylvester proved a general identity from which Euler's Pentagonal Numbers Theorem followed as a special case. It is normally the experience that one discovers a q-series identity first and later obtains a combinatorial explanation or proof. In this situation, Sylvester had obtained his general identity combinatorially but was unable to provide a generating function proof. So he challenged the mathematical community to find such a proof. His long time friend Cayley responded to the challenge and produced a beautiful generating function proof. This has a connection with the Rogers–Ramanujan identities as we shall soon see. George Andrews has remarked that Sylvester's identity holds the distinction of being the first q-series identity whose first proof was purely combinatorial!

Sylvester stayed in Baltimore for only 8 years. In 1883 he was elected to the Savillian Professorship of Geometry at Oxford University. He occupied this position until 1893 when his health began to decline. He died in 1897 in England at the age of 83. Ramanujan was then a ten year old boy in India.

Connection with Ramanujan's Work As is well known, the Rogers–Ramanujan identities are among the deepest and most elegant in the theory of partitions and q-series. Ramanujan discovered these identities in India prior to his departure to England but was unable to prove them. The British mathematician L.J. Rogers had proved them in 1894. Rogers later gave a second proof which was considerably simpler in detail than his first. After Ramanujan arrived in England and studied Rogers' paper in 1917, he was able to give his own proof. When Rogers and Ramanujan corresponded in 1917, it turned out that Rogers had a third proof which was remarkably similar to Ramanujan's. In view of the importance of these identities, Professor G.H. Hardy arranged for Rogers and Ramanujan to write a joint paper in which both their proofs were presented. This paper appeared in The Proceedings of The Cambridge Philosophical Society in 1919. The paper opens with Hardy's introductory remarks. Rogers' proof is given in Section 1, and Ramanujan's in Section 2. It is interesting that both Rogers' and Ramanujan's proofs make use of an identity

which can be considered as the next case beyond the general identity that Sylvester found. But the method employed by Rogers and Ramanujan is not the combinatorial method of Sylvester, but the method underlying Cayley's proof of Sylvester's identity.

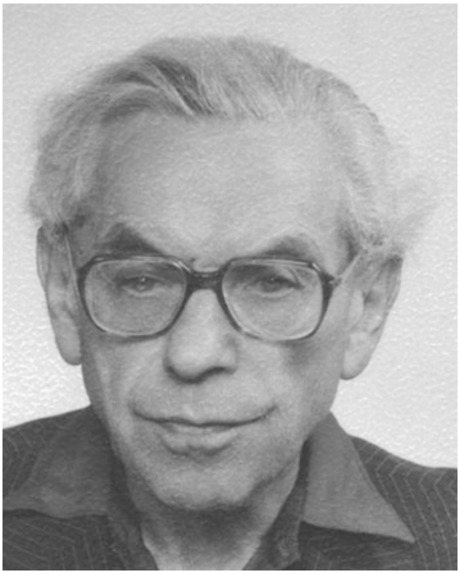
In the 1960s Basil Gordon obtained an elegant and important generalisation of the Rogers–Ramanujan identities. One way to prove Gordon's generalisation is to establish a suitable extension of Sylvester's identity.

It should be noted that I. Schur, the German mathematician, independently proved the Rogers–Ramanujan identities in 1917. Owing to World War I, he was cut off from England and so was unaware of the work of Rogers and Ramanujan. Schur's proof is very much combinatorial and in fact goes along the lines of Franklin's proof of Euler's Pentagonal Numbers Theorem. Although several proofs of the Rogers–Ramanujan identities are known today, none are simple. For a long time, it was an open question whether a combinatorial bijective proof of the Rogers–Ramanujan identities can be found. In 1980, Garsia and Milne found a bijective proof, but even this is very complicated. The key idea in the Garsia–Milne proof is based on Schur's combinatorial constructions which can be traced back to the techniques put forth by Sylvester and his students. Thus we see that Sylvester's paper has an influence in current research.

There are other instances where Sylvester's work intersects with that of Ramanujan, as Andrews has pointed out in his New Zealand Lectures. In Ramanujan's Lost Notebook there are several identities which have partition interpretations, but Ramanujan, as was his custom, never stated their combinatorial significance. Ramanujan was more an analyst than a number theorist. So he preferred the analytic form of his identities and did not make their number-theoretic or combinatorial forms explicit. Andrews has shown that many of Ramanujan's identities have elegant partition interpretations and that by using Durfee squares one can not only prove some of these identities but extend them as well.

George Andrews, who has explained the significance of many of Sylvester's ideas on partitions in the context of current research, has said often that Sylvester's paper of 1882 is still worthy of deeper study. On the subject of partitions Sylvester said: "Partitions constitute the sphere in which analysis lives, moves and has its being; and no power of language can exaggerate or paint too forcibly the importance of this till recently almost neglected, but vast, subtle and universally permeating, element of algebraical thought and expression." The theory of partitions continues to be active area of research because of the wealth of ideas handed down by mathematicians of eminence like Euler, Sylvester and Ramanujan.

Chapter 7
Erdös and Ramanujan: Legends of Twentieth Century Mathematics



Erdös

In September 1996, Professor Paul Erdös, one of the mathematical legends of this century, died while attending a conference in Warsaw, Poland. His death at the age of 83 marked the end of a great era, for Erdös was not only an outstanding mathematician but a very kind and generous human being who encouraged hundreds of mathematicians over the decades, especially young aspirants to the

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subject. Many including me owe their careers to him. He was without doubt the most prolific mathematician of this century, having written more than 1000 papers, a significant proportion of them being joint papers. Even in a mathematical world which is used to geniuses and their idiosyncracies, Erdös was considered an unusual phenomenon and was viewed with awe and adoration just as Ramanujan evoked surprise and admiration. And like Ramanujan's mathematics, the contributions of Erdös will continue to inspire and influence research in the decades ahead. In this article I will discuss some remarkable aspects of the life and contributions of Professor Erdös. In doing so, I will explain connections with the work of Ramanujan and other Indian mathematicians. Finally, I will also describe how he was a strong influence in my career, first when I was a student, next when I was at MATSCIENCE, and more recently during my tenure in Florida.

Eternally on the Move Paul Erdös was unique in many ways. Born in Hungary in April 1913, he was a member of the Hungarian Academy of Sciences. But he did not have a job or any regular position. He was constantly on the move, criss-crossing the globe several times during a year visiting one university after the other giving lectures. He seldom stayed at one place for more than two weeks except perhaps in his native Hungary where he returned periodically between his travels. And he did this every year for the past half a century or more! Can you imagine what it would take to maintain this hectic schedule each year for so long? To be in constant demand at universities throughout the world, one should not only be an unending source of new ideas, but also have the ability to interact with persons of varying tastes and abilities. Erdös was superbly suited to this task. This is what kept his furious productivity going till the very end. In a long and distinguished career starting in 1931, Erdös wrote well over 1000 papers making fundamental contributions to many branches of mathematics, most notably, Number Theory, Combinatorics, Analysis and Geometry.

The Erdös Number There is a well-known joke in mathematical circles that any mathematician who has written a joint paper should have an Erdös number. Erdös himself has Erdös number 0. A person who has written a joint paper with him has Erdös number 1. A collaborator of a collaborator of Erdös, has Erdös number 2, and so on. Since Erdös had so many collaborators, the joke is that if a mathematician has a joint paper, it should be possible to trace a connection with Erdös! For instance, Einstein has Erdös number 2, because Einstein collaborated with Ernst Straus (my thesis advisor) who had several joint papers with Erdös. It has been suggested that for someone who has written n joint papers with Erdös, the Erdös number should be $1/n$ instead of 1. If that were the case, then I am proud that my Erdös number is $1/5$.

Awards Naturally, anyone with such a long and distinguished career must have received several awards and recognitions. In 1952, Erdös was the recipient of the

prestigious Cole Prize of The American Mathematical Society for his fundamental contributions to Number Theory. In 1983, he was awarded the Wolf Prize for his lifelong contributions to mathematics, and he joined the ranks of other illustrious winners of this prize like Kolmogorov and André Weil. He was elected member of the National Academy of Sciences of USA and also elected Foreign Member of The Royal Society. He is also the recipient of numerous honorary doctorates from universities around the world.

Almost a Saint What did Erdös do with his income and prize money? Erdös who was a bachelor all his life was wedded to mathematics which he pursued with a passion. He was not unlike Ramanujan who also had few interests outside of mathematics. Erdös had no desire for any material possessions and so was saintly in his attitude towards life. He often used to say that property was a nuisance. During his visits to universities and institutes of higher learning, he was paid honoraria for his lectures. After keeping what was necessary to pay for his travel and stay expenses, he would give away the remaining amount either in the form of donations to educational organisations or as prizes for solutions to mathematical problems he posed. I should emphasise that Erdös was, without doubt, the greatest problem proposer in history. During his lectures worldwide, he posed several problems and offered prize money ranging from $50 to $1000, depending on the difficulty of the problem. This was one way in which he spotted and encouraged budding mathematicians. In talking about Ramanujan it has often been mentioned that his greatness was not only due to the remarkable results he proved, but also due the many important questions that arose from his work. Similarly, Erdös will not only be remembered for the multitude of theorems he proved, but also for the numerous problems he raised.

Even though Erdös did not seek fame or fortune and had no desires for earthly possessions, he still had to attend to matters relating to his travels and make sure that there was enough money for his prizes. In this regard, Ronald Graham of Lucent Technologies (formerly known as AT&T Bell Labs.) has been of immense help to him. Graham, who was one of Erdös' discoveries, is a leader in the field of Combinatorics. He was Past President of The American Mathematical Society. For the past several years, Graham was coordinating Erdös' itinerary and handling his accounts. A question that Erdös was often asked was whether he would be able to pay if all his problems are solved simultaneously. To this he humourously responded as follows: "What will happen to the strongest bank if all its creditors decide to withdraw their money at the same time? That is more likely to happen than my problems being solved all at once!"

Inspired by Ramanujan Erdös began his illustrious career as a mathematician with a paper in 1932 on prime numbers. Interestingly, it was through this paper that he first became aware of Ramanujan's work. Naturally, Ramanujan was a strong influence and inspiration for him from then on.

Prime numbers have held the fascination of mathematicians from the golden age of Greece to the present day. A well-known assertion known as Bertrand's postulate states that for any positive number n, there is always a prime between n and $2n$. This fact was first proved by the Russian mathematician Chebyshev. Erdös, who was 18 years old in 1931, found a beautiful new proof. Upon seeing this, the Hungarian mathematician Kalmar pointed out that Ramanujan had found a similar proof. Erdös then looked up the collected works of Ramanujan and was fascinated by the startling beauty of the results of the Indian genius. In the decades that followed, Erdös' research on arithmetical functions and in number theory had many connections with Ramanujan's work, and I will discuss this briefly now.

In 1918, Ramanujan published a long paper on *highly composite numbers*. These are numbers n which have more divisors than all numbers m less than n. Ramanujan established several fundamental properties about highly composite numbers. Erdös got interested in the problem and improved some of Ramanujan's results. One improvement he obtained was a lower bound for $C(x)$, the number of highly composite numbers up to x. Ramanujan had actually a second manuscript on highly composite numbers, but this was not published because of paper shortage in England during World War I. During Ramanujan's centennial in 1987, when the Lost Notebook was brought out in printed form, certain other unpublished hand-written manuscripts of Ramanujan including the second manuscript on highly composite numbers were also printed. In the second issue of The Ramanujan Journal which will appear in April 1997, this second paper of Ramanujan on highly composite numbers will be published along with notes and explanations by the French mathematicians Jean-Louis Nicolas and Guy Robin.

Perhaps the most significant connection between the work of Erdös and Ramanujan relates to the creation of Probabilistic Number Theory, and I describe this next.

Probabilistic Number Theory During his brief stay in England, Ramanujan wrote several fundamental papers, some jointly with G.H. Hardy of Cambridge University. In one such joint paper published in 1917, Hardy and Ramanujan explored the properties of *round numbers* by discussing the number of prime factors of integers. What is surprising here is that, although prime numbers have been studied since Greek antiquity, Hardy and Ramanujan were the first to systematically discuss the number of prime factors among integers. One of the fundamental results they showed in this paper was that almost all integers n have about loglog n prime factors. Roughly speaking, round numbers n are those which have substantially more prime factors. The true significance of this result was realised only in 1934 when the Hungarian mathematician Paul Turán, a close friend of Erdös, found a new and simpler proof. This proof indicated that important probabilistic principles might lie at the heart of the Hardy–Ramanujan result. In 1939, during a lecture in Princeton, the great mathematician Mark Kac described some possible new applications of Probability Theory, in particular to problems related to the Hardy–Ramanujan results on the number of prime factors. But in order to carry out these ideas of Kac, various auxiliary results from Number Theory were required. Fortunately, Erdös was in the audience. As soon as Kac finished his lecture, Erdös walked up to him and showed

how *sieve methods* from Number Theory could be used to complete the arguments of Kac. Thus the celebrated Erdös–Kac theorem was proved, and with this, Probabilistic Number Theory was born. To quote Erdös on this event: "During the lecture of Kac, I realised that by Brun's (sieve) method, I could deduce the conjectures of Kac by his methods. After the lecture we immediately got together. He was not an expert in Number Theory, and I knew very little of Probability. Thus neither of us completely understood what the other was doing, but we realised that our joint work will yield the (Erdös—Kac) theorem. This collaboration is a good example to show that two brains are better than one, since neither of us could have done the work alone." Probabilistic Number Theory is an active area of research today, and the two-volume book by Peter Elliott describes many of the major advances starting with the work of Hardy–Ramanujan and the impetus given by Erdös–Kac.

Partitions The most famous joint paper of Hardy and Ramanujan is the one in which they obtained an asymptotic formula for $p(n)$, the number of partitions of an integer n. This paper is important not only because of the results proved but also owing to the techniques employed. It was in this paper that the *circle method* was introduced. This powerful analytic technique is now the standard tool used in a wide variety of problems in Additive Number Theory, and its genesis may be traced to a formula that Ramanujan communicated to Hardy in a letter written from India in 1913. Erdös was fascinated by this work of Hardy and Ramanujan. In a paper published in the Annals of Mathematics in 1942, he was able to show that the leading term in the Hardy–Ramanujan asymptotic formula follows by purely elementary methods from an identity due to Euler. Erdös is of course the all time champion of elementary methods. In fact his most celebrated contribution is the elementary proof of the Prime Number Theorem.

The Prime Number Theorem One of the most important results in Number Theory is the Prime Number Theorem conjectured independently by Legendre and Gauss. This theorem states that the number of primes below a given magnitude x is asymptotically $x/\log x$. It was finally proved towards the end of the last century simultaneously and independently by the French mathematicians Hadamard and de la Vallée Poussin using Complex Variable Theory. All attempts to obtain an elementary proof failed, and so it was widely believed that an elementary proof was not possible. In fact, Professor G.H. Hardy went to the extent of declaring that if an elementary proof of this theorem was ever to be found, then our text books in the subject would have to be rewritten because this would change our views drastically. In 1949, to the complete surprise of everyone, Paul Erdös and Atle Selberg obtained such an elementary proof. For this and their other achievements, Selberg was given the Fields Medal in 1950 at the International Congress of Mathematicians in Harvard, and Erdös was awarded the Cole Prize of The American Mathematical Society in 1952.

Interaction with Indian Mathematicians Erdös had always regretted that he never met Ramanujan. Similarly, he was disappointed not to have had the chance to

meet the famous Indian number theorist, Sivasankaranarayana Pillai. Erdös got interested in a problem on iterates of the Euler function that Pillai had worked on and was looking forward to meeting him during the International Congress of Mathematicians in Harvard in 1950. Unfortunately S.S. Pillai died in an air crash in Cairo on the way to the conference. But during his long career, Erdös interacted with several Indian mathematicians. He was especially proud of his collaboration with the noted mathematician Sarvadaman Chowla who passed away in December 1995. During the past two decades I had the pleasure and privilege of interacting with Professor Erdös very closely, and I discuss my association with him next.

Chancellor's Fellowship My research interests in Number Theory started at the age of 16 when as a student entering the B.Sc. class at Vivekananda College, I discovered some new properties of the Fibonacci numbers. My father, Professor Alladi Ramakrishnan, realised that my interest deserved encouragement, and so he contacted various number theorists around the world for their advice and opinion. Some of these mathematicians suggested that I should contact Paul Erdös whose life's mission was to spot and encourage mathematically talented youngsters. Fortunately, an opportunity for me to meet Erdös soon arose.

In December 1974, there was a conference at the Indian Statistical Institute in Calcutta to which Erdös was invited as a principal speaker. It turned out that one of my undergraduate papers was accepted for presentation at that conference. However, I could not attend that meeting because of my college exams. My father who was also participating in that conference, presented my work on my behalf. Erdös, who spoke to my father after the lecture, expressed interest in my work and said that he would like to meet me. He was on his way to Australia then, and so it was natural for him to stop in Madras on his way from Calcutta. So Erdös came to Madras for a few days to give lectures at MATSCIENCE and at the Ramanujan Institute. During this period I was with him all the time, attending his lectures and discussing mathematics with him. There were three immediate consequences for me as a result of his visit. First, he wrote a letter in early January 1975 to Professor Ernst Straus of UCLA (a former assistant to Albert Einstein) recommending me. Before the end of January, I received a letter from UCLA offering me the Chancellor's Fellowship to do a Ph.D. there. Next, he took my paper on higher-order divisors with him to Australia. In Sydney he met George Szekeres, a long time friend of his from Hungary who was one of the Editors of the Journal of the Australian Mathematical Society. Erdös spoke to Szekeres about my paper on higher-order divisors and arranged for its publication there. Finally, Erdös liked my work on the arithmetical function $A(n)$, the sum of the prime factors of n. Starting from our first meeting in Madras, we had a several letters of correspondence on this function. This resulted in two joint papers which appeared in the Pacific Journal of Mathematics.

Erdös was right. Professor Straus was the perfect thesis advisor for me. The three years I spent at UCLA (1975–1978) were among my happiest and most productive, and I am thankful to Erdös and Straus for that.

Visits to MATSCIENCE After my Ph.D., I spent a few years in the United States to gain post doctoral experience, first at the University of Michigan in Ann Arbor,

next at The Institute for Advanced Study in Princeton, and finally at the University of Texas in Austin. Erdös was among those who wrote letters of recommendation that helped me secure these positions. In the summer of 1981, between my stays at Princeton and Ann Arbor, I joined Matscience, and within a few weeks, Erdös visited Madras. I organised a Number Theory conference for Matscience in Mysore with Erdös as the principal participant. It was during this conference that he expressed a desire to see the beautiful hill station Ootacamund, Ooty as it is well known, and hoped that next conference might be there. As he desired, in 1984, I was able to arrange another Matscience Number Theory Conference, this time in Ooty for his 70th birthday. Several mathematicians from Hungary and the USA participated in this conference. Interestingly, he received the Wolf prize just before this conference; so he came to India soon after the prize was awarded. After the conference, he spent some time at Matscience as Ramanujan Visiting Professor.

Visits to Florida It was also during this visit to India that he advised me that it was best that I return to the United States for the sake of my career. Once again, he was instrumental in making the transition possible. In 1985, a position opened up at the University of Florida in Gainesville. Erdös had a long association with Florida dating back to the 1960s. It turned out that I was visiting the University of Hawaii that year, and so it was easy to get me to Florida for a lecture. Primarily owing to his support, I was offered a permanent position there which I took up in December 1986. Fortunately for me, Erdös visited Florida every year as he had been doing since the 1970s when his distinguished friend Stan Ulam was a Graduate Research Professor there. Somehow in his worldwide travels, like migrating birds, Erdös always managed to hover around the isotherm 70 °F! So he visited Calgary in the summers, California in the winters, and Florida in February/March.

The Ramanujan Centennial In December 1987, leading mathematicians from around the world gathered in India to pay homage to Ramanujan on the occasion of his centennial. Anna University asked me to organise a one-day Number Theory Symposium in Madras as part of the centennial celebrations. Naturally I invited Erdös to participate in the symposium. Unfortunately he could not come due to an eye surgery. But he kindly sent a paper entitled “Ramanujan and I” which appeared in the proceedings of that symposium published by Springer Verlag which I edited. In that paper he described how Ramanujan was a source of inspiration for him. In fact, the description of the connections between the work of Erdös and Ramanujan that I have given in this article is based upon the paper of Erdös in the Anna University Conference Proceedings.

The Ramanujan Journal Two years ago, when the idea to start the Ramanujan Journal was put forth, Erdös was very supportive. When he was invited to serve on the Editorial Board, he agreed very graciously. The Ramanujan Journal will be launched in January 1997 by Kluwer Academic Publishers of the Netherlands. Had he been alive, he would have been delighted to see the first issue appear. But before he died, he contributed a paper to the first volume, written jointly with Carl

Pomerance and Andras Sárközy. Thus he blessed the Ramanujan Journal with his contribution. In January and April of 1998, as part of Volume II, the Ramanujan Journal will bring out two issues in memory of Prof. Erdös. These issues will also be made available in hard cover (in book form) for those who may wish to purchase them separately. By publishing the Erdös memorial issues in the Ramanujan Journal, we are paying a fitting tribute to Erdös and Ramanujan, both of whom are legends of twentieth century mathematics.

Chapter 8
C.G.J. Jacobi: Algorist par-excellence

Jacobi

C.G.J. Jacobi (1804–1851) was one of the greatest mathematicians in history. His magnum opus, the *Fundamenta Nova*, a monumental treatise on the elliptic and theta functions, was one of the most influential mathematical books in the first half of the nineteenth century. In his treatment of theta functions, Jacobi demonstrated his superior skills in the manipulation of infinite series and products, inspired by

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his illustrious predecessor Leonhard Euler. One of the most striking features of Ramanujan was his remarkable ability in handling infinite series, products and continued fractions, and therefore he is compared to Jacobi. Ramanujan had his own theory of elliptic and theta functions and recorded several beautiful theorems on these in his notebooks. Some were rediscoveries of earlier work, but the majority of the theorems were new and original. It is still a mystery how Ramanujan arrived at these results on elliptic and theta functions without any formal training, and whether he had exposure to earlier work on the subject. Here I will describe the life and some of the major contributions of Jacobi and comment upon the relationship with Ramanujan's work. In preparing this article I have made use of E.T. Bell's classic *Men of Mathematics* for biographical details on Jacobi and Hardy's *Twelve lectures on Ramanujan* and Bruce Berndt's *Ramanujan's Notebooks—Part III* with regard to Ramanujan's methods, results and their significance. I have also benefited from several helpful comments by Professors Richard Askey and Bruce Berndt.

Education Carl Gustav Jacob Jacobi was born in Potsdam, Prussia, on 10 December 1804 as the second son of a prosperous banker, the first son being called Moritz; a story about Moritz is at the end of this article. Jacobi also had another brother and a sister. Jacobi's first teacher was one of his maternal uncles who taught him classics and mathematics that helped Jacobi's entry into the Potsdam Gymnasium in 1816. Jacobi soon demonstrated that he had a universal mind, and with high praise from the rector of the Gymnasium, he left in 1821 for the University of Berlin where he was until 1825. In Berlin he divided his time equally between philosophy and mathematics, but it was the latter which attracted him most. His Ph.D. thesis of 1825 on partial fractions was a good one, but nothing in comparison with the outstanding work that was soon to follow; the thesis however gave clear indications of his manipulative skills in algebra.

Elliptic and Theta Functions Soon after his Ph.D., Jacobi launched himself with great enthusiasm into the study of elliptic and theta functions. Theta functions are bilateral infinite series in two variables z and q, a typical one having the exponent of z running through all the integers and the exponent of q running through the squares of the integers. They arise naturally in statistical mechanics and as solutions to the heat equation. Suitable ratios of theta functions give rise to elliptic functions which are doubly periodic. Their importance in the vast subject of complex variable theory cannot be overestimated. The theory of elliptic functions was an important testing ground for the discovery and improvement of general theorems in the theory of functions of a complex variable, and the two subjects compliment and supplement one another. Thus elliptic functions attracted the attention of the very best mathematicians of the nineteenth century such as Abel, Gauss, Jacobi, Legendre and Weierstrass.

The Fundamenta Nova Jacobi applied elliptic and theta functions to the theory of numbers in a significant way. His mastery over infinite series and products is

best seen in his treatment of the theta functions. He showed that every theta function admits a product representation. This is the fundamental *Jacobi Triple Product Identity*, a special case of which yielded Euler's pentagonal numbers theorem on partitions. By studying the fourth power of a certain theta function, Jacobi not only gave another proof of Lagrange's famous theorem that every positive integer is a sum of four squares, but he was able to get a formula for the number of representations as well. Jacobi's work excited the admiration of Gauss, and this was no mean achievement, for Gauss was not one to be impressed easily. This recognition by Gauss led Jacobi to be promoted from the post of a lecturer to that of an assistant professor at the University of Konigsberg in 1827. This was quite an achievement for a young man of twenty three, and many of his colleagues who were superceded, resented the appointment. But in 1829 when he came out with the *Fundamenta Nova*, his masterpiece on elliptic and theta functions, all criticism subsided, and the colleagues congratulated the brilliant Jacobi.

Hard Times In 1832, Jacobi's father died. Until this time, Jacobi need not have worked for a job. With money left by his father, the prosperity continued for another eight years until the fortune collapsed in 1840. Jacobi was now in need of money to support his family. These financial difficulties however had no negative effect on his mathematical research, and his productivity was unabated. Gauss who had been observing Jacobi's prodigious output with obvious interest, was concerned that these hardships might eventually end Jacobi's mathematical career. Fortunately in 1842, the King of Prussia agreed to support Jacobi with a substantial allowance. Unfortunately Jacobi fell ill soon after, but the king generously advised Jacobi to take rest in Italy and agreed to continue the allowance during the absence. In the process of getting cured of his illness, Jacobi came into contact with a physician who suggested that he should try his hand in politics. Jacobi succumbed to this ill advice. His short stunt in politics was a disaster, and he incurred the displeasure of the king who terminated the allowance in 1849. Jacobi who was then 45 years old was the father of seven small children and penniless. Hearing of his plight, the Austrian king invited Jacobi to Vienna and offered to support him. It was then that Alexander von Humboldt intervened and talked to the King of Prussia and persuaded him to restore the allowance. Jacobi continued his mathematical investigations vigourously until his death in 1851 due to small pox.

Vast Contributions The subject of elliptic and theta functions was the first great work of Jacobi. He made significant contributions to many other areas. One of Jacobi's great glories was that he extended the work of Hamilton in dynamics. The Hamilton–Jacobi equation plays a fundamental role in mechanics. In algebra, Jacobi cast the theory of determinants in such a clear form that it became accessible to beginning college students. Of much higher order of originality was his discovery of Abelian functions. Elliptic functions which are doubly periodic arise out of the inversion of certain integrals, called elliptic integrals. These integrals derive their name because they come up in the computation of the arc lengths of ellipses. What Jacobi realised was that this theory could be extended by inverting certain other

integrals called Abelian integrals, leading to Abelian functions, the simplest case being certain functions of two variables but having four periods. More generally, Abelian functions in n variables have $2n$ periods. E.T. Bell says: "This discovery was to nineteenth century analysis what Columbus' discovery of America was to fifteenth century geography."

Comparisons Jacobi's mathematical development was in some sense parallel to that of his great rival Abel who was two years older. Both Abel and Jacobi were inspired by the study of elliptic functions and integrals, but their approaches were different. Abel along with Gauss seemed to have been more determined about what to do because of conceptual ideas, whereas Jacobi was more guided by formulas like Euler. E.T. Bell says: "For sheer manipulative ability in tangled algebra, Euler and Jacobi have no rival, unless it be the Indian mathematical genius Srinivasa Ramanujan of our own century." Jacobi who had a very objective mind was not at all jealous of Abel's contributions. In fact Jacobi said of Abel's work: "It is above my own praises as it is above my works." Similarly Ramanujan had spontaneous praise for L.J. Rogers when he first came across papers of the British mathematician wherein certain identities he had stated in letters to Hardy were actually proved.

Ramanujan's Letter and Reactions In his two letters to Hardy of 1913, Ramanujan communicated a variety of theorems on infinite series, products, continued fractions, integrals and special evaluations. There were several results related to elliptic and theta functions. Some of the results had been proved already. For instance, there was a formula expressing the truncation of the Gaussian integral in terms of a continued fraction. This formula is originally due to Laplace but was first rigourously proved by Jacobi. But there were several theorems in the letters that were startlingly new and original revealing Ramanujan's mastery over infinite processes. When Hardy showed the letters to his colleague Littlewood, it was this aspect that prompted the latter to say that Ramanujan was at least a Jacobi. Hardy and Littlewood knew that Ramanujan had no formal mathematical training, which explained why many of the theorems were rediscoveries. But in the case of elliptic and theta functions where the theory is quite sophisticated, it was difficult for Hardy to believe that Ramanujan had arrived at his results without any exposure to earlier work. Hardy knew that *Carr's Synopsis* was the first book to open up Ramanujan's mind. This book is an outcome of Carr's tutorial notes and contains a collection of results listed sequentially with only hints of proofs every now and then. Carr's book does have a short section on elliptic integrals. However, in the case of elliptic functions, Hardy suspected that Ramanujan had perhaps seen Greenhill's book which was available at the Madras University Library at that time. The treatment of elliptic functions in Greenhill's book is rather unusual because double periodicity is discussed only in the later chapters. Since Ramanujan never made use of double periodicity, Hardy felt that he was perhaps influenced by Greenhill. This mystery could easily have been resolved if Hardy had asked Ramanujan about Greenhill's book or other sources during their frequent discussions in Cambridge. In the preface to his twelve lectures on Ramanujan, Hardy is very apologetic that he did not

clear up this mystery when he was in an ideal position to do so. But he explained the situation as follows: "Ramanujan was quite willing to give a straight answer to a question and not in the least inclined to make a mystery of his achievements… But it seemed to me ridiculous to ask him how he proved this theorem or that when he was showing me about half a dozen new ones every day."

Ramanujan's Methods Chapters 16–21 of Ramanujan's second notebook contain the bulk of his observations on elliptic and theta functions along with other q-series identities. The edited form of these chapters may now be found in *Ramanujan's Notebooks—Part III* by Bruce Berndt. So we are in a much better position today to understand Ramanujan's methods which were highly original.

Ramanujan's notation for theta functions was quite unique. He wrote his basic theta function as a bilateral series in variables a and b where the exponents of both variables were certain triangular numbers. This is equivalent to Jacobi's representation by change of variables. But Ramanujan's version led him to certain new results on theta functions.

Ramanujan deduced the Jacobi triple product identity in an unusual way, as a limiting case of his famous one-psi-one summation. An identity one level above the triple product identity is the *quintuple product identity*, which can actually be traced back to Weierstrass. Most proofs of the quintuple product identity make use of the triple product identity. In his notebooks Ramanujan had observed certain interesting special cases of the quintuple product identity, but this identity appears in full form in Ramanujan's Lost Notebook. Master as he was over infinite processes and limits, Ramanujan failed to notice that the Rogers–Ramanujan identities would follow as a limiting case of an identity recorded as Entry 7 in Chap. 16 of his second notebook.

Modular Equations In the area of modular equations, Ramanujan was unrivalled. He established more modular equations than Gauss, Jacobi and all others combined. Today modular equations are once again a topic of great interest especially after they have been utilised to calculate several million digits of π. Ramanujan's methods in elliptic and theta functions are so different from his predecessors, and many of his results are so new and original, Professor Berndt believes that Ramanujan might have developed his theory without a trace of external influence.

Immortality I conclude this article with a humourous anecdote reported by E.T. Bell: "The name Jacobi appears frequently in the sciences, not always meaning the same man. In the 1840s one very notorious Jacobi, M.H., had a comparatively obscure brother C.G.J., whose reputation was but a tithe of M.H.'s. Today the situation is reversed: C.G.J. is immortal while M.H. is rapidly receding into obscurity… M.H. achieved his fame as the founder of… galvanoplastics. C.G.J.'s… much higher reputation is based on mathematics. During his lifetime, the mathematician was always being confused with his more famous brother, or worse, being congratulated for the involuntary kinship. At last C.G.J. could stand it no longer. "Pardon me, beautiful lady," he retorted to an enthusiastic admirer of

M.H., who complimented him on having so distinguished a brother, “but *I* am my brother.” On other occasions C.G.J. would blurt out, “I am not *his* brother, he is *mine*.”

Both Jacobi and Ramanujan have been immortalised by the importance of their discoveries and by their ingenious methods.

Chapter 9
Evariste Galois: Founder of Group Theory

Galois

G.H. Hardy of Cambridge University, Ramanujan's mentor, said that the real tragedy of Ramanujan's life was not his early death, but the fact that in his most formative years, the Indian genius spent much time proving results which were rediscoveries of past work due to lack of proper guidance in India. Hardy argued that in mathematics especially, the most brilliant work is done when one is very young, so if Ramanujan had lived longer, he would have proved more theorems, but might not

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have produced work of higher quality. Hardy cited as an example Evariste Galois, founder of Group Theory, who met an untimely end in a duel at the age of twenty. Hardy pointed out that Galois, like mathematicians Abel who died at 22 and Riemann at 40, had done his best work by then. Even the great Gauss, the Prince of Mathematicians, who lived a full life, made most of his discoveries in his teens and spent the rest of his life polishing up his results for suitable presentation. In this article I shall describe the tragic all too brief life of Galois and the path-breaking discoveries he made. I shall also make a comparison with the life and contributions of Ramanujan. Finally I shall describe the impact that Group Theory has made in various fields and the present state of its research. In doing so, I will talk briefly about the contributions of my distinguished colleague Professor John G. Thompson on the University of Florida, arguably the greatest group theorist since Galois!

For biographical details pertaining to the life of Galois, I have relied on *MacTutor History*, and for the development of Group Theory, I have consulted an article by O'Connor and Robertson. With regard to the work of Ramanujan, I profited from the article entitled *Ramanujan's association with radicals in India* by Berndt, Chan and Zhang that appeared in 1997 in the American Mathematical Monthly.

Life of Galois Evariste Galois was born on 25 October 1811 in Bourg La Reine near Paris, France. His parents were well educated, but there is no indication of mathematical talent in his family. Galois' father was well known in the community and was elected mayor of his township in 1815. This period in France was quite tumultous and indeed saw rapid changes in leadership. Napoleon who was at the height of his power in 1811 was thrown out in 1815 after the defeat at Waterloo. The political instability that followed was to have a devasting effect on the young Galois.

Galois' performance in elementary school at the Lycee in Louis-le-Grand was very good, and in fact he won several prizes. However in 1826 he was asked to repeat a year because his work in rhetoric was not up to the standard. Age 16 was a turning point in his career because it was then that he took his first mathematics course under M. Vernier. Like Ramanujan he became engrossed in mathematics to such an extent that the Director of Studies wrote: "It is the passion for mathematics that dominates him. I think it would be best if his parents would allow him to study nothing but this. He is wasting his time here and does nothing but torment his teachers and overwhelm himself with punishment."

The École Polytechnique was the premier university in Paris, but Galois failed the entrance exam in 1828. So he was back at the Louis-le-Grand and enrolled in a mathematics course by Loius Richard. It was at this time that he read Legendre's classic treatise on Geometry. As Richard wrote "This student works only in the highest realms of mathematics." In April 1829 Galois published his first paper on continued fractions (a favourite topic of Ramanujan's) in the Annales de Mathématiques. Shortly thereafter he started submitting articles on algebraic solutions to equations, a topic for which he would soon contribute revolutionary and far reaching ideas.

Unfortunately, as a result of politically based conspiracy, Galois' father committed suicide on 2 July 1829. This tragedy that struck the family had a telling effect

on the young Galois. For one thing, Galois failed the entrance exam to the École Polytechnique the second time he took it shortly after his father's death. This forced him to enter the École Normale which was an annex to the Louis-le-Grand. His total immersion in mathematics like Ramanujan's cost him in his performance in other subjects. His literature examiner wrote: "This is the only student who answered me poorly, he knows absolutely nothing. I was told that this student has an extraordinary capacity for mathematics. This astonishes me greatly, because after the examination, I believed him to have but little intelligence." Whatever be the opinion of the literature examiner, Galois at that time wrote a beautiful mathematical paper entitled *On the condition that an equation be soluble by radicals* that was being considered for the Grand Prize by the Academy. Unfortunately, the paper was in the possession of Fourier who died in 30 April; the paper was subsequently never found and so was not considered for the prize after all!

Sophie Germain (known now for a major contribution in connection with Fermat's Last Theorem) wrote a letter to a mathematical friend describing Galois' situation: "... the death of M. Fourier has been too much for this student Galois ... He has been expelled from the École Normale. He is without money ... They say that he will go completely mad. I fear that this is true."

To make matters worse, in the midst of this mental depression following Fourier's death and that of his father, Galois got involved in political controversies that were raging in France. He was imprisoned for open demonstrations and even attempted to commit suicide in prison. In March 1832, a cholera epidemic swept through Paris, and all prisoners including Galois were transferred to the Pension Sieur Faultrier. There he fell in love with Stephanie-Felice du Montel, daughter of the resident physician. Although he was released shortly thereafter, his freedom was short lived. Once again, for political reasons, he was imprisoned, but this time he was to fight a duel to get out. Galois was aware that he was fighting a superior adversary and that most likely he would be killed in the duel. So on the night before the duel, he wrote a letter to a friend outlining the wonderful new ideas he had in connection with the solvability of algebraic equations. Galois died in the duel on 31 May 1832, at the tender age of 20. The reasons for the duel are not entirely clear, but Stephanie's name appears as a marginal note in the manuscript that Galois wrote the night before he was killed. Fortunately his letter was preserved. The revolutionary mathematical ideas in this letter led to the birth of Group Theory, a central branch of mathematics with important applications in several other fields as well.

The Birth and Growth of Group Theory The subject of Group Theory deals with symmetries in general such as those that arise in geometry and in solutions to polynomial equations. Although certain key properties associated with groups can be traced to earlier mathematicians, it was only with the work of Galois that the concept of a group crystallised and concrete applications of the concept emerged. Three different streams that gave rise to group theory were (i) geometry at the beginning of the 19th century, (ii) number theory at the end of the 18th century, and (iii) the theory of algebraic equations at the end of the 18th century leading to the study of permutations.

Since the study of geometry goes back to antiquity, it is natural to ask what was the reason for the emergence of the group concept in the 19th century via geometry. During the 19th century, a mathematical revolution was taking place with the emergence of non-Euclidean geometry and synthetic geometry. Suddenly, instead of just angles and lengths dominating the discussions, invariances under transformations were the key to geometrical study, and indeed, this eventually led to the study of transformation groups.

Euler, the most prolific mathematician in history, systematically studied remainder arithmetic in number theory during the mid 18th century. This was subsequently called modular arithmetic by Gauss who took it several steps further. In the work of Euler and Gauss in number theory, group theoretical properties (as we know today) associated with remainders were crucial, but neither of them formulated the group concept in generality.

We all learn the quadratic formula in school, namely the formula which gives the solutions to the general quadratic equation. We are told that there are similar but more complicated formula for roots of the general cubic and quartic equations, but we are not given these formulas. The French mathematician Lagrange wanted to find out why the cubic and quartic equations could be solved algebraically. In this connection, permutations were first studied by Lagrange in a classic paper of 1770 on the theory of algebraic equations. Although the beginnings of permutation groups can be seen in this work, Lagrange does not discuss the general group concept at all.

The first person to claim that there is no general quintic formula, namely, the non-existence of a formula to solve all quintics, was Ruffini in 1799. Ruffini's work on quintics was based on Lagrange's permutation approach but had gaps in his reasoning. It was Abel in 1824 who gave the first complete proof of the insolvability of general quintics. It is here that Galois enters the picture.

Galois in 1831 was the first to really understand that the algebraic solvability of a polynomial equation was intimately related to the group structure of certain permutations associated with the equation. In his now famous letter of 1832 written on the eve of his death, Galois had demonstrated by the study of groups that there is no general formula that will give the roots of all polynomials of degree n, when n is at least five. Galois' work was not known until Liouville published it posthumously in 1846. But even then, Liouville failed to grasp the group concept that was the key to Galois' work. The understanding of the general group concept, and the realisation that it was the basis of Galois' work, came only in the second half of the 19th century. Thus like Ramanujan, Galois was much ahead of his time, and a full grasp of his ideas came only decades later.

By 1872, Group Theory was becoming the centre stage of mathematics because Felix Klein of Gottingen in his famous *Erlangen Programme* called for the group theoretic classification of geometry. Group Theory really came of age with the publication of the book *Theory of Groups of Finite Order* by Burnside in 1897. Also the two volume book *Lehrbuch der Algebra* by Weber in 1895 and 96 became a standard text. These books influenced the next generation of mathematicians to make group theory as perhaps the most major single branch in 20th century mathematics.

With advanced and abstract mathematics playing a prominent role in the sciences during the 20th century, group theory became a crucial tool outside of mathematics as well, for instance, in quantum mechanics in physics and crystal structure in chemistry.

Group Theory Today In the modern era, the most prominent figure in Group Theory is John Griggs Thompson. Born in Ottawa, Kansas, in 1932, Thompson entered Yale University in the early 1950s to earn his BA in Theology with the desire to become a Presbyterian minister. His interest in mathematics was sparked when his roommate drew his attention to George Gamow's book *One, two, three, infinity*. From then on, the rest is history.

The call of mathematics was too strong to resist. Thompson changed his major and received a Bachelors in Mathematics at Yale in 1955 and moved to the University of Chicago for his Ph.D. His Ph.D. thesis of 1959 was a masterpiece. It was not just an extension of known techniques, but full of new and powerful ideas that soon led to major developments in group theory. The most sensational of these was the resolution of a long standing conjecture that all finite groups with an odd number of elements are solvable. Thompson proved this in collaboration with Walter Feit, and their 253 page proof in 1963 occupied one entire issue of the Pacific Journal of Mathematics! For this magnum opus, Feit and Thompson received the Cole Prize of the American Mathematical Society in 1966. Thompson continued establishing further fundamental results and in 1970 was awarded the Fields Medal, the highest prize in mathematics equivalent in prestige to the Nobel Prize, at the International Congress of Mathematicians in Nice. That year he was also appointed Rouse Ball Professor of Mathematics at Cambridge University, a position that he held until 1993, when he moved over to The University of Florida as Graduate Research Professor.

Thompson's name is also closely associated with one of the monumental achievements of the 20th century, namely, the classification of finite simple groups. In any field of study, one tries to understand complex objects in terms of those simpler in structure. For example, in number theory, we try to understand properties of integers by decomposition into prime factors. In group theory, finite simple groups are basic building blocks. Starting at the time of Thompson's thesis, group theory leapt into prominence as the mathematical topic undergoing the most rapid development. The main reason for this was that it became clear that the classification of all finite simple groups was now realisable, and not just a dream. The classification was completed only in the early 1980s as a collective effort of many noted mathematicians, and Thompson's ideas were crucial in this effort.

In the past few years, Thompson has been working on, and made major contributions to, the famous Inverse Galois Problem which has remained unsolved. Certain special types of groups that Galois investigated in connection with the algebraic solvability of polynomial equations are called Galois groups today. The Inverse Galois Problem states that given an arbitrary finite group, one can produce a setting in which the given group is the Galois group of a certain polynomial.

Since Thompson's productivity over the years at the highest level has remained unabated, he has received numerous awards and recognitions in a steady stream.

He was elected to the US National Academy of Sciences in 1971 and made Fellow of the Royal Society in 1979. He was awarded the Sylvester Medal of the Royal Society in 1987 and the Wolf Prize of Israel in 1992. That year he also received the Poincaré Golden Medal by the Académie des Sciences, Paris. This medal is awarded only on exceptional occasions, the two previous recipients being Jaques Hadamard (1962) and Pierre Deligne (1974). And on 1 December 2000, Thompson was awarded the National Medal of Science by President Clinton for his lifelong contributions to mathematics. I had the honour of representing the University of Florida at the Medals Ceremony and the pleasure of seeing Thompson receiving the medal. We at the University of Florida feel privileged to have Thompson as a colleague and to know personally one of the greatest mathematicians of the 20th century.

Ramanujan and Radicals Ramanujan's interest in algebraic solutions to polynomial equations can be seen by his work on radicals. A radical is an expression involving combinations of various nth roots of integers. When one solves a polynomial equation algebraically, such as with the quadratic formula, one expresses the solution in terms of radicals. Out of the 58 problems that Ramanujan submitted to the Journal of the Indian Mathematical Society, ten of them involve equalities between radicals. During Ramanujan's time, especially among British mathematicians, establishing identities involving radicals was quite common. Ramanujan investigated radicals in connection with the study of class invariants. The German mathematician Weber had studied class invariants extensively, but Ramanujan found an astonishing number of new ones. It was only after his arrival in Cambridge that Ramanujan knew of Weber's work. Ramanujan used class invariants to find excellent approximations to π, as well as determine explicitly values of theta functions at certain points. It is still a puzzle as to what methods Ramanujan used to compute these class invariants. He has left no clues in his notebooks. Since Weber's methods were highly algebraic, it is unlikely that Ramanujan pursued such techniques. Thus, as Berndt, Chan and Zhang say in their paper on Ramanujan and radicals, "Ramanujan's ideas still remain hidden behind an opaque curtain."

Conclusion In summary there are many similarities in the life stories of Galois and Ramanujan. Both had plenty of obstacles in their way. Undaunted by these, both continued to pursue mathematics with a passion and made outstanding discoveries marked with supreme originality. The tragedy is that both died very young, and we can only contemplate what more they might have accomplished had they lived longer. Finally, what a remarkable coincidence, that both communicated their most important findings in letters just before their death. Galois letter gave birth to Group Theory, and Ramanujan's last letter to Hardy created the subject of mock theta functions. We should be thankful that these geniuses have left behind ideas for succeeding generations to ponder on and develop and that their legacy remains strong even in this new millennium!

Chapter 10
Leonhard Euler: Most Prolific Mathematician in History

Euler

Leonhard Euler is one of the greatest mathematicians of all time, on par with Gauss, Newton and Archimedes. Euler's fundamental contributions have influenced almost every branch of mathematics. He was without doubt the most prolific mathematician in history. Indeed, even the blindness that physically handicapped him in his later years did not diminish his furious productivity. Euler was a master of infinite series and products. Eric Temple Bell in his famous book *Men of Mathematics*

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says that with regard to the ease and brilliance with which Euler manipulated infinite processes, only Ramanujan rivalled him. Similarly, G.H. Hardy of Cambridge University compared Ramanujan to Euler and Jacobi for sheer manipulative ability. In this article I will describe the life and some of the pioneering contributions of Euler and make comparisons with the discoveries and methods of Ramanujan. For biographical details, I have relied on the book *Euler—the master of us all* by William Dunham, published by the Mathematical Association of America (1999). I also profited from comments by Professors George Andrews, Bruce Berndt and Alexander Berkovich.

Early Life and Education Euler was born in 1707 near Basel, Switzerland. He was a precocious youth blessed with a gift of languages and extraordinary memory. Dunham says "Euler… carried in his head an assortment of curious information, including orations, poems, and lists of prime powers. He was a fabulous mental calculator, able to perform arithmetical computations without benefit of pencil or paper. These uncommon talents would serve him well in later life." Ramanujan too had a prodigious memory and as a boy could recite the Sanskrit roots (*Atmanepada* and *Parasmepada*) and several digits in the decimal expansions of numbers like e, π and the square root of 2.

Upon joining the University of Basel at the age of 14, Euler came in contact with its most famous professor, Johann Bernoulli (1667–1748). The professor was known to be proud and arrogant, and did not hesitate to put down the ideas of others. Interestingly, Johann Bernoulli was much impressed with Euler and permitted the young lad to have frequent discussions with him. Very soon, the role of mentor and pupil was reversed because as Euler matured, it was he who was showing the way for Johann Bernoulli. Indeed Johann Bernoulli, not easily given to compliments, wrote the following generous lines to Euler: "I present higher analysis to you in its childhood, but you are bringing it to a man's estate." In the case of Ramanujan, Hardy says emphatically that a single professor like him was insufficient for so fertile a pupil and that Ramanujan was showing him about half a dozen new results every day.

Euler's university education was not restricted to mathematics. He studied the history of law and obtained a masters degree in philosophy. Then he entered divinity school to become a priest and registered in the faculty of theology. This required him to study Greek and Hebrew. But the call of mathematics was too strong, and Euler simply could not find time to progress in other subjects. So he quit divinity school to pursue what he loved most, mathematics. In the case of Ramanujan, mathematics was an obsession to the extent that he neglected other subjects totally and so did not even earn a college degree. Dropping out of divinity school enabled Euler to concentrate fully on mathematics. At the age of 20 he gained recognition in an international scientific competition for his analysis of the placement of masts on a sailing ship, unusual for someone who had lived all his life in land-locked Switzerland! Such accomplishments would serve him well in the future.

Position at St. Petersburg In 1725, Daniel Bernoulli, son of Johann, was appointed to a position in mathematics at the newly formed St. Petersburg Academy

in Russia, and he invited Euler the next year to join. The only opening at that time was in physiology/medicine, and since Euler did not have any other options, he accepted that position. Happily for Euler, by the time he arrived in St. Petersburg in 1727, he found out that he had been reassigned to physics. Euler stayed at the home of Daniel Bernoulli, and the two engaged in elaborate discussion of physics and mathematics that eventually strongly influenced the development of these subjects in Europe. In 1733 Daniel Bernoulli returned to Switzerland for an academic position. While this was a loss to Euler, it also enabled him to occupy the position in mathematics at the St. Petersburg Academy that Daniel Bernoulli had vacated.

The setting in St. Petersburg was ideal for Euler. He had plenty of time to pursue his mathematical research and also was at the disposal of the state which paid his salary. As a scientific consultant to the government, he prepared maps, advised the Russian navy, and even tested designs of fire engines.

It was in St. Petersburg that he had his first great triumph by solving the notorious *Basel Problem*. The question was to determine the exact value of the convergent infinite series obtained by summing the reciprocals of the squares of the positive integers. The problem was posed by Pietro Mengoli in 1644, but it was Jakob Bernoulli, Johann's brother, who brought it to the attention of the broader mathematical community in 1699. The problem had stumped several noted mathematicians, and anyone solving it was sure to gain instant recognition. Not only was Euler's method of attack brilliant, but the value surprised everyone: it turned out to be one sixth of the square of π.

With the *Basel Problem* behind him, Euler began writing research papers at a furious pace. Nearly half of the papers published by the St. Petersburg Academy turned out to be his work. But Russia got into a serious political turmoil with the death of Empress Catherine I. So Euler accepted an offer from Prussia's Fredrick the Great and made a move in 1741 to take up a position at the newly revitalized Berlin Academy.

Work in Berlin Euler was at the Berlin Academy for a quarter century. It was the middle phase of his mathematical career. It was during this period that he published his two greatest works, *Introductio in analysin infinitorum*, a 1748 text on functions, and *Institutiones calculi differentialis*, a 1755 volume on differential calculus. It was also in Berlin that he discovered the famous *Euler's identity* giving the value of the exponential function in terms of the trigonometric functions sine and cosine. A special case of this identity yields the remarkable formula that the number e raised to the power $i\pi$ plus 1 equals 0. Thus this formula combines the most fundamental numbers in mathematics: e (the natural base of the logarithms), i (the imaginary square root of -1), π (the ratio of the circumference of a circle to its diameter), 0 (the identity for addition), and 1, the identity for multiplication.

In Berlin Euler was asked to provide instruction to Princess Anhalt Dessau. The result was a multi-volume masterpiece in exposition. These *Letters to a German Princess* from Euler gained international attention and turned out to be Euler's most read work.

Return to St. Petersburg While in Berlin, Euler maintained a cordial relationship with the St. Petersburg Academy and published numerous papers in their journal. He continued to receive a stipend from St. Petersburg even during the period when Russian forces entered Berlin during the Seven Year War. It turned out that over a period of time, the relationship between Euler and Fredrick the Great worsened because Euler spoke German and business in Prussia was conducted in French. Voltaire gained the favour of Fredrick the Great, and consequently, the relationship between Euler and Voltaire suffered. Sadly, even though Euler had brought considerable fame to the Berlin Academy, he was forced to leave. Fortunately, the political climate in Russia had improved considerably under Catherine the Great, and the St. Petersburg Academy was ready in 1766 to welcome back Euler, the world's greatest mathematician.

During the reign of Catherine the Great, art and culture flourished. She had immense regard for Euler and treated him almost like royalty. She set aside a fully furnished house for Euler and his dependents and even donated one of her own cooks to run his kitchen. She consulted Euler on various matters and valued his wisdom and counsel. The following humourous incident is a good illustration. Catherine had invited the French philosopher Denis Diderot, but in due course actually got fed up with his efforts to convert her courtiers to atheism. So she summoned Euler to handle the adamant philosopher. Euler knew that Diderot had no understanding of mathematics. So he challenged Diderot to a contest. Right at the start, Euler walked up to Diderot and said in a tone of perfect conviction, *"Sir, $a +$ the nth power of b divided by n equals x, hence God exists. Reply."* Diderot was flabbergasted and speechless. Owing to the humiliation, he suffered in the presence of the full court, Diderot asked for Catherine's permission to return to France.

Final Years in St. Petersburg Unfortunately, a few years after his return to Russia, two tragedies struck Euler. The vision problems that began in St. Petersburg prior to moving to Berlin had worsened considerably, and in 1771 Euler became totally blind. Also in 1773, his wife Katerina died. These would be crushing blows to anyone and virtually would stop one's productivity. But Euler was a remarkable individual. Undeterred by these setbacks, his mind continued to pour out mathematical theorems in rapid succession. In fact during this period of blindness, he wrote a 775 page treatise on Algebra! Three years after the death of his wife, Euler married her half sister, who was with him until his death on 18 September 1783. Euler was in full control of his mathematical powers until the final moments of his last day. That morning, after spending some time with his grandchildren, Euler took up some mathematical questions concerning the flight of balloons, spurred by the exciting news of the Montgolfier brothers' balloon flight over Paris. In the afternoon, he made some calculations on the behaviour of Uranus. The peculiar orbit of this planet attracted Euler. Later generation astronomers used Euler's preliminary calculations to seek and discover the next planet Neptune. If Euler had lived longer, he probably would have predicted mathematically the existence of a planet beyond Uranus. Unfortunately he suffered a massive hemorrhage that day and died immediately.

I will now describe briefly a sample of Euler's fundamental contributions and connect some of these to Ramanujan's discoveries.

Perfect Numbers The Pythagoreans were fascinated by integers with special properties. They defined a perfect number to be a positive integer which is equal to the sum of all its divisors other than itself. Thus $6 = 1 + 2 + 3$ and $28 = 1 + 2 + 4 + 7 + 14$ are perfect. Euclid showed that if a number m is one less than 2 raised to the power of a given prime p, and if m is prime, then m multiplied by 2 to the power $p - 1$ is a perfect number. While this is a procedure to construct perfect numbers, the difficulty is that there is no known algorithm to show when such numbers m will be prime. Such primes m are called Mersenne primes, and we do not know whether there are infinitely many Mersenne primes. Euclid's procedure constructs only even perfect numbers, and it does not confirm that there are infinitely many even perfect numbers. The main significant result after Euclid on perfect numbers was due to Euler who showed that ALL even perfect numbers are obtained by the procedure Euclid gave. Thus with Euler's and Euclid's results, we have a precise description of the structure of even perfect numbers. Strangely, there is not a single example of an odd perfect number, and it is conjectured that no odd number can be perfect. But this ancient simple to state problem has remained unsolved!

Euler's Constant Euler was a master in the manipulation of infinite series and products. It is now a well-known fact to any student of calculus that the harmonic series, namely, the infinite series obtained by summing the reciprocals of the positive integers, diverges. In the days of Euler, there was no clear understanding of the notions of convergence and divergence. The rigorous approach to convergence and divergence came with Cauchy. In Euler's works there are statements on convergence and divergence which would either be taken as incorrect or incomplete statements by present day rigour. However, like Ramanujan, Euler knew what he was stating and how to deal with such situations. Indeed, as in the case of identities in Ramanujan's notebooks, if Euler's statements are interpreted properly, then they are not only correct, but very significant. The situation with harmonic series highlights this point.

Euler asserts that the sum of the harmonic series equals the natural logarithm of infinity plus a quantity that is nearly a constant. What Euler is saying is that if the harmonic series is truncated at a point x, that is if the first x terms of the harmonic series are summed, then the value of this is approximated by the natural logarithm of x. More precisely, he says that if the natural logarithm of x is subtracted from the sum of the first x terms, then the difference tends to a constant as x approaches infinity. This constant is known today as Euler's constant. By the study of the harmonic series and related sums, Euler systematically developed an important method of summation of series which is now known as Euler–Maclaurin summation. With regard to Euler's constant, a famous unsolved problem is to decide whether it is a rational or irrational number.

Ramanujan had his own theory of summing infinite series. To an infinite series convergent or not, he would associate a value he called *the constant of the series*. For instance, for the infinite series obtained by summing the positive integers, the value he attached was $-1/12$. When he showed this to mathematics teachers in India, they viewed these claims as incorrect and outrageous. However, it turns out that there is great significance underlying this claim. One of the most fundamental

objects in analytic number theory is the Riemann zeta function. The sum of the positive integers can be formally viewed as a the representation of the zeta function at -1. It turns out that if one uses complex variable theory and evaluates the value of the zeta function at -1 through a process called analytic continuation, then this value is indeed $-1/12$. It is amazing that Ramanujan who (according to Hardy) did not possess a clear understanding of complex function theory, came up with this value $-1/12$ by his theory of constants for infinite series. Ramanujan's theory of constants has connections with the Euler–Maclaurin summation.

Preponderence of Primes Euclid proved that there are infinitely many primes. Euclid's proof of this and his proof of the irrationality of the square root of 2 using the *reductio ed absurdum* method are true gems in mathematics. The Greeks however do not seem to have raised the question as to how dense the primes are among the integers. Utilizing the divergence of the harmonic series like the logarithm of x, Euler showed that the sum of the reciprocals of the primes also diverges. Indeed Euler made the assertion that the sum of the reciprocals of the primes equals the logarithm of the logarithm of infinity. What Euler meant was that if the sum of the reciprocals of all primes up to a given number x is considered, then this sum is approximated by the value $\log\log x$. Thus the primes are dense enough among the integers that the sum of their reciprocals diverges. This approximation of the sum to $\log\log x$ by Euler is the first estimate of the preponderence of primes. This estimate and other evidence led nineteenth century mathematicians Legendre and Gauss to conjecture *The Prime Number Theorem*, namely, that the number of primes below a given magnitude x divided by $x/\log x$ tends to 1 as x tends to infinity. The Prime Number Theorem was proved only at the end of the nineteenth century using techniques from complex variable theory.

Reciprocals of the Cubes As was mentioned above, Euler was the one who evaluated the sum of reciprocals of the squares of the positive integers. In the context of this result and the estimate on the divergence of the harmonic series, the natural question to ask is what the value of the sum of the reciprocals of the cubes of the positive integers is. We now view this as the value of the Riemann zeta function at 3 and denote it by $\zeta(3)$. Euler conjectured a value for this in terms of the logarithm of 2 and π, but this did not lead anywhere. Like Euler's constant gamma, $\zeta(3)$ was conjectured to be irrational, but this was proved only in 1978 by Roger Apery. Apery's proof is truly remarkable and led to a resurgence of activity in the area of irrationality. There is charming account of Apery's proof by Alfred van der Poorten in his article "A proof that Euler missed—Apery's proof of the irrationality of zeta 3" which appeared in the Mathematical Intelligencer in 1980. The exact value of $\zeta(3)$ is still not known, and there is not even a conjectured value.

With regard to the Riemann zeta function, Euler showed that the values at the even positive integers $2k$, namely $\zeta(2k)$, are rational multiples of π raised to the power $2k$. This formula was rediscovered by Ramanujan. As for the values at odd integers greater than 1, namely $\zeta(2k+1)$, it is conjectured that these are all irrational, but the only case for which this has been proved is that of $\zeta(3)$.

Euler's Function Pierre Fermat (1601–1665) is considered one of the founding fathers of number theory. A question that interested Fermat was the multiplicative structure of integers that are not divisible by a given prime number p. Fermat discovered that when any positive integer a less than p is raised to the power $p-1$ and then divided by p, the remainder is always 1, no matter what the values of p and a are. This is sometimes referred to as Fermat's Little Theorem. Euler generalized this beautifully by considering the number of integers less than any given integer n and having no common factors with n. This is known as Euler's function (of n). This function is very fundamental not only in number theory, but in other branches as well. For instance, in group theory, one learns that the number of generators of a cyclic group of order n is the Euler function of n.

Fermat's Last Theorem As is well known, after studying the work of Diophantus of Alexandria, Fermat asserted that for exponents n which are at least 3, it is impossible to find three non-zero integers x, y, z such that the nth powers of x and y will add up to the nth power of z. This assertion of Fermat attained fame and noteriety because he claimed to posses a "truly remarkable proof", and the margin of the book in which he made this assertion was too small to contain the proof! This became known as Fermat's Last Theorem and was solved only recently by Andrew Wiles and Richard Taylor. Great as Fermat was, often there were gaps in the proofs of his claims. Euler not only improved on many of Fermat's theorems (such as the case with Euler's function described above), but also filled the gaps in several of Fermat's proofs. For instance, Fermat showed how to prove the "Last Theorem" for the exponent $n=4$ by the (now famous) method of infinite descent and claimed that the same method could be used to settle the case $n=3$. But it was Euler who supplied the arguments for this case.

The Taxi-Cab Equation Ramanujan was mainly interested in equations which had solutions, and especially in providing algorithms or formulas for the solutions. This could be a reason why he was not interested in Fermat's Last Theorem. The famous Ramanujan taxi-cab number 1729 is an example of an integer which is expressible as a sum of cubes in two different ways (1729 is the sum of the cubes of 12 and 1, as well as the sum of the cubes of 10 and 9) and indeed is the smallest such example. One could view this problem as a mild variation of the one Fermat considered, namely, we are considering four non-zero integers x, y, z, w such that the sum of the cubes of x and y equals the sum of the cubes of z and w. Amazingly, addition of an extra variable w yields a solution. Euler had provided a general formula for solving this taxi-cab equation, but in one of Ramanujan's notebooks, there is also a general formula for the solution to this equation which is some ways is more elegant than Euler's.

Euler's Identity As was mentioned earlier, while in Berlin, Euler made the fundamental discovery that the exponential of an imaginary angle $i\theta$ could be expressed as $\cos\theta + i\sin\theta$. This is called Euler's identity and is pivotal to the understanding of complex numbers. Every complex number z, represented by a point in the complex

plane, can either be written as $x + iy$ in terms of its Cartesian coordinates x and y or be assigned its polar coordinates r and θ. Euler's identity provides a fundamental link between the Cartesian and polar representations via the exponential function. The consequences are amazing. For example, from this it follows that if two complex numbers a and b are multiplied to yield a complex number c, then the way to locate c in the complex plane is to multiply the distances of a and b from the origin, and sum the angles a and b make with the x axis. Among other things, this shows that the number 1 has n nth roots (two square roots, three cube roots, etc.), produces a method of constructing these nth roots, and reveals the beautiful group structure that exists among the nth roots of unity.

In talking about Ramanujan's methods, Hardy says that the Indian genius did not really have a knowledge of the theory of functions of a complex variable and indeed was not familiar with Cauchy's theorem at all. However, Ramanujan was very familiar with properties of complex numbers. In fact, he studied the sum of the powers of the primitive nth roots of unity and established several interesting properties for them. These sums are now called Ramanujan sums. For example, the sum of all the primitive nth roots of unity takes only the values 1, 0 or -1 and is equal to the famous Moebius function of n.

The Euler Characteristic One of Euler's most remarkable observations is that for any three-dimensional bounded convex polytope (an object bounded by planes), the number of vertices minus the number of edges plus the number of faces is always equal to 2. For example, a cube has 8 vertices, 12 edges, and 6 faces, and $8 - 12 + 6 = 2$. A pyramid has 5 vertices, 8 edges, and 5 faces, and $5 - 8 + 5 = 2$. This observation of Euler is not restricted only to polytopes, but can be extended to bounded convex objects in three-dimensional space. For example, consider the earth viewed as a sphere, which is not a polytope (but topologically equivalent to a cube), and take as vertices any finite collection of points on it, say the north and south poles. So in this example we have two vertices. For edges we may take any finite collection of longitudes. Let us take for instance the Greenwich Meridian and the International Date Line. So the number of edges is also 2. Now in this example there would be only two faces, namely the surfaces bounded by these two longitudes, the eastern and western hemispheres. Here again $2 - 2 + 2 = 2$. More generally, for a convex bounded object in n-dimensional space, one may study the number of vertices (zero-dimensional subspaces on the boundary of the object) minus the number of edges (one-dimensional subspaces on the boundary) plus the number of faces (two-dimensional subspaces on the boundary) minus the number of three-dimensional subspaces making up the boundary, and so on. This is the Euler characteristic of the object in n-dimensional space. The Euler characteristic is a fundamental invariant in the field of topology.

The Euler Line While studying Euclidean geometry in high school, we learn several fundamental facts about triangles. Among these are: (i) The medians of a triangle are concurrent. The point of concurrency, the centroid, is the centre of gravity of the triangle assuming that the triangular plate is uniformly thick and is made

of uniformly dense material. (ii) The perpendicular bisectors of the three sides of a triangle are concurrent. The point of this concurrency, called the circumcentre, is the centre of the circle which passes through the three vertices of the triangle. (iii) The three perpendiculars from each of the vertices to the opposite sides are concurrent. The point of this concurrency is called the orthocentre. Any good high school student must be fascinated by such lovely properties. Euler showed the remarkable fact that for all triangles, the orthocentre, the centroid, and the circumcentre, are all collinear. The common line on which these three points lie is called the Euler line. Euler earned his position in the Geometry Hall of Fame by this single result!

Partitions The theory of partitions founded by Euler has today become a central topic of research owing to its interactions with many fields within and outside of mathematics. A partition of a positive integer n is a representation of that integer a sum of positive integers, two such representations being considered the same if they differ only in the order of the parts. Thus $4+2+1$ and $4+1+2$ represent the same partition of 7. Following Euler, we write the parts of a partition in descending order. The number of partitions of n is denoted by $p(n)$. There are 7 partitions of 5 given by 5, $4+1$, $3+2$, $3+1+1$, $2+2+1$, $2+1+1+1$, and $1+1+1+1+1$. Thus $p(5) = 7$. Euler was interested in partitions because by means of their generating function representations, he could prove beautiful results. These generating functions are infinite power series, and Euler was a master in manipulating them. One fundamental result he proved using generating functions was that the number of partitions of any integer into distinct (non-repeating parts) equals the number of partitions of that integer into odd parts (that could repeat). This is called Euler's theorem. From the above example we see that 5, $4+1$, and $3+2$ are the three partitions of 5 into distinct parts, while 5, $3+1+1$, and $1+1+1+1+1$ are the three partitions of 5 into odd parts. Another of Euler's results is the celebrated pentagonal numbers theorem. Here Euler shows that the reciprocal of the generating function of $p(n)$ can be expanded into a power series whose coefficients, except when the powers are pentagonal numbers, are zero; when the powers are pentagonal numbers, the coefficients are 1 and -1 with a certain definite pattern. Euler established such results by formal operations on power series generating functions and did not prove these combinatorially even though the results had combinatorial interpretation because they pertained to partitions. Similarly, Ramanujan too stated his results as identities for power series and did not emphasize the partition theoretic or combinatorial significance. For example, the celebrated Rogers–Ramanujan identities are stated by both Rogers and Ramanujan as infinite series having product representations. Neither Rogers nor Ramanujan stated the combinatorial version of these identities. Such an interpretation was due MacMahon and Schur, who observed that *For* $i = 1, 2$, *the number of partitions of an integer into parts differing by at least* 2, *and least part being at least* i, *equals the number of partitions of that integer into parts which when divided by* 5 *leave remainder* i *or* $5 - i$. This Rogers–Ramanujan partition theorem can be considered as a theorem which is the next step beyond Euler's in a certain sense (pointed out by Schur), but it is much deeper and more complicated to prove. Because of the combinatorial interpretation, Schur was able to discover

the next level partition result. Today the theory of Rogers–Ramanujan-type partition identities connecting partitions whose parts satisfy difference conditions with those partitions whose parts satisfy congruence conditions has become an important area of research because of its interactions with the theory of modular forms, Lie algebras and physics.

In summary, Euler who had an unbelievably productive career spanning more than half a century, made fundamental contributions to several branches of mathematics. Here I selected only a few of his results that I could explain in lay terms or could connect to Ramanujan's work. In the case of the theory of partitions, it is no exaggeration to say that while Euler was its illustrious founder, it was under Ramanujan's magic touch that the subject underwent a glorious transformation. E.T. Bell in his classic *Men of Mathematics* begins his description of the life of Euler with a quote of Arago, "Euler could calculate with no apparent effort as men breathe or as eagles sustain themselves in the wind." I think the same could be said of the Indian mathematical genius Srinivasa Ramanujan.

Chapter 11
G.H. Hardy: Ramanujan's Mentor

Hardy

G.H. Hardy, a towering figure in analysis and number theory, had written several important research papers and influential textbooks on these subjects. When Ramanujan wanted to get the opinion of British mathematicians to evaluate his discoveries which lay at the interface between analysis and number theory, it was only natural that he chose to write to Hardy. Actually Ramanujan communicated his remarkable findings to several British mathematicians, but it was only Hardy who responded.

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Realising that Ramanujan was a genius of the first magnitude who would profit immensely by contact with professional research mathematicians, Hardy invited Ramanujan to Cambridge University, England. The rest is history. The collaboration between Hardy and Ramanujan, the influence they had on each other, and the impact their work had over mathematicians of their generation and those succeeding them was immense. In this article, I will describe briefly some aspects of the life and contributions of Hardy with specific emphasis on his collaboration with Ramanujan and the impact this has had. For some facts about Hardy's life, I have relied on the article by R.J. Wilson in the book *Cambridge Scientific Minds* edited by Harman and Mitton.

Early Life and Education G.H. Hardy was born on 7 February 1877 in Cranleigh, in Surrey, England, as the elder of two children. His father was a school teacher at Cranleigh School, and his mother a Senior Mistress at a training college. Although both parents were mathematically minded, they did not attend university owing to financial reasons.

Both Hardy and his sister were encouraged to read good literature and discover things for themselves. Hardy showed precocity with arithmetic and passed examinations with distinction in mathematics and Latin at Cranleigh School. He then secured a scholarship at Winchester College but was not satisfied with the education there. Although he intended initially to go to Oxford University from Winchester, he changed his mind and joined Trinity College at Cambridge University in 1896.

Cambridge Education At Cambridge, Hardy started to train for the famous Tripos exams. These exams were undoubtedly among of the most severe mathematical tests. Although Hardy was successful in solving the Tripos problems, he hated these exams and what they stood for. In fact Hardy was much irritated with his Tripos coach Dr. Webb. Fortunately, the Director of Studies put Hardy under the care of the applied mathematician Augustus Love, who asked Hardy to read Camille Jordan's wonderful treatise *Cours d'Analyse*. Hardy says "My eyes were opened by Professor Love who taught me for a few terms and gave me the first serious conception of analysis. I shall never forget the astonishment with which I read the remarkable work (of Jordan)..."

Just as Hardy detested the formality of the Tripos, Ramanujan was uninspired by the dull college curriculum. Whereas Hardy was inspired by Jordan's rigorous treatment of analysis which influenced his own style of research, Ramanujan was influenced by Carr's synopsis which according to his Indian biographers was a treasure house of delightful formulas that awakened his genius. Unlike Ramanujan, who dropped out of college, Hardy held on to be successful at the Tripos exams in 1900 and was in fact placed in the first division of First Class. Hardy was elected to a prize fellowship at Trinity College and in 1901 won the prestigious Smith's prize along with the physicist James Jeans.

Influential Papers and Books It was in 1900 that Hardy began to publish his mathematical discoveries. His first paper on definite integrals appeared in the Messenger of Mathematics. Hardy subsequently published more than sixty papers in the

theory of integration over the next three decades. Between 1905 and 1915 Hardy wrote four books in the Cambridge Mathematical Tracts series on *The Integration of Functions of a Single Variable (1905), Orders of Infinity (1910), The General Theory of Dirichlet Series (with Marcel Riesz in 1915)*, and what was to become his most well-known book, *A Course in Pure Mathematics (1908)*. During the post Newtonian era, although there were many notable mathematical contributions from England, British mathematics had taken a backseat compared to the phenomenal contributions from mainland Europe. Hardy and a few other British mathematicians, notably J.E. Littlewood, Hardy's collaborator in Cambridge, were leading the British revival in mathematics. Hardy was an especially polished and prolific writer and had the greatest effect in leading this resurgence of British mathematics. His books and papers had a powerful influence not only on mathematicians in England but overseas as well. Indeed, Hardy's book *Orders of Infinity* attracted Ramanujan who was intrigued by the problem of determining the number of primes below a given magnitude, and it was this that prompted Ramanujan to write to Hardy.

Ramanujan's Letters The two letters Ramanujan wrote to Hardy in 1913 are considered to be among the greatest in mathematical history. The letters not only contained a fantastic collection of spectacular formulae, but also profoundly influenced the mathematical careers of both Hardy and Ramanujan. Hardy's initial reaction on seeing the letters was that Ramanujan was a fraud because many of the formulas were known, some were incorrect, and there were no hints of proofs. But then there were several astonishingly beautiful formulas that were correct and very deep. Only a mathematician of the highest class could write them down. So on second thought Hardy concluded that it was more probable that Ramanujan was a genius and unlikely that he was a fraud because no one but a true genius could have the imagination to invent such formulae. The great philosopher Bertrand Russell says that one evening in Trinity College he found the usually placid Hardy in a wild state of excitement talking about a new Euler or Jacobi from India! Hardy was convinced that Ramanujan was wasting his time in India rediscovering past work and would profit immensely by coming into contact with professional mathematicians. He thus invited Ramanujan to Cambridge University to work with him. Although Ramanujan's mother initially resisted this, she eventually realised that she should not stand in the way of her son's progress and so gave Ramanujan permission to go to England.

The Hardy–Ramanujan Interaction Ramanujan spent only a few years in England (1914–1919), and although he was ill quite often in this brief period, he wrote several fundamental papers, some jointly with Hardy. These papers have had a lasting influence on various branches of mathematics and led to surprising connections between fields considered quite distinct. When Ramanujan was well, he met Hardy almost every day and was showing his mentor a dozen new formulas at every meeting. It was difficult for Hardy to keep up with Ramanujan's progress and leaps of imagination. Indeed Hardy admitted that a single supervisor was inadequate for so fertile a pupil! Yet, the raw genius of Ramanujan combined with the scholarship of

Hardy produced magnificent results. I shall now briefly describe two very significant joint papers they wrote.

Formula for Partitions In one of the letters to Hardy in 1913, Ramanujan gave a formula for the coefficients of a certain series expansion of an infinite product which suggested that there ought to be a similar exact formula for $p(n)$, the number of partitions of a positive integer n. Hardy felt that this claim of Ramanujan concerning an exact formula for $p(n)$ in terms of continuous functions was too good to be true but was convinced that it would be possible to construct an asymptotic series expansion. Asymptotic formulas are like approximate formulas. By an ingenious and intricate calculation involving the singularities of the generating function of $p(n)$, Hardy and Ramanujan obtained an asymptotic formula which when calculated up to a certain number of terms yielded a value that differed from $p(n)$ by no more than the fourth root of n. Since $p(n)$ is an integer, it is clear that its value is the nearest integer to what is given by the series. The proof of this formula required sophisticated tools from complex variable theory, and here Hardy's mastery over analytic methods was crucial. The series Hardy and Ramanujan obtained was genuinely an asymptotic series in the sense that when summed to infinity it diverges. Subsequently, Hans Rademacher noticed that by making a very mild but important change, namely, by replacing the exponential functions by hyperbolic functions, the Hardy–Ramanujan asymptotic series could in fact be converted to an infinite series that converges to $p(n)$. Actually in the 1913 letter to Hardy, Ramanujan used hyperbolic functions to claim an exact formula for a related problem. So Ramanujan was indeed correct in surmising that a similar exact formula would exist for $p(n)$. Professor Atle Selberg of the Institute for Advanced Study in Princeton, one the greatest living mathematicians, said during his talk at the Ramanujan Centennial in Madras in 1987, that the exact formula of Rademacher was indeed more natural than the Hardy–Ramanujan asymptotic formula. Selberg discovered this formula on his own but did not publish this once he found out that Rademacher had obtained it earlier. According to Selberg, even though Ramanujan correctly claimed the existence of an exact formula for $p(n)$, out of respect for his mentor Hardy, he settled for less, namely the asymptotic formula. In any case, the Hardy–Ramanujan asymptotic formula gave rise to a very powerful analytic method to evaluate the coefficients of series arising in a wide class of problems in additive number theory. This *circle method*, originally due to Hardy–Ramanujan, was subsequently developed by Hardy and Littlewood and others, and is one of the most widely applicable methods today.

Round Numbers Hardy and Ramanujan were interested in a mathematical explanation of the phenomenon *round numbers are rare*, where loosely speaking, a number is round if it is composed of a large number of relatively small prime factors. So they went about studying the behaviour of $\omega(n)$, the number of prime factors of a positive integer n. It is a curious fact of history that although prime numbers have been studied since Greek antiquity, it was only in the beginning of the twentieth century with the work of Hardy and Ramanujan that the number of prime factors of n was studied systematically. Hardy and Ramanujan showed that on average $\omega(n)$ was

about $\log \log n$ in size and that this average also indicated the size of $\omega(n)$ almost always. This was somewhat surprising since for most number-theoretic functions, the average is not a true indication of the size, because the average is often influenced by large values which occur infrequently. In the case of $\omega(n)$, the large values had no influence on the average because they occurred with very low frequency. Thus they explained that round numbers, which are numbers for which $\omega(n)$ is large, are rare. The true significance of the Hardy–Ramanujan observation on round numbers was not realised until a few decades later, when Paul Turán, Paul Erdös and Mark Kac showed the probabilistic underpinnings of this result. Indeed, with the Hardy–Ramanujan work on round numbers as the foundation point, and with the beautiful superstructure built on it by Turan, Erdös and Kac, the subject of Probabilistic Number Theory was born and is a very active field of research today.

Honours for Ramanujan Hardy was convinced that since Ramanujan's work was phenomenal, he deserved to be honoured by the Royal Society. But time was running out. Ramanujan was getting ill more often and was frequently in and out of nursing homes. At one point Ramanujan was so depressed that he attempted suicide. There is a story that when the police wanted to question Ramanujan on the attempted suicide, Hardy intervened and convinced the police not to go through with this by telling them that Ramanujan was a Fellow of the Royal Society (FRS), which he was not at that time! Concerned about the continued illness, Hardy arranged for Ramanujan to return to India where he felt the care would be better with the help of kith and kin. But Hardy wanted Ramanujan to get the honours before his return to India for two reasons. First, he felt that the honours would boost Ramanujan's spirits and could have a positive effect on his health. Second, Hardy feared that Ramanujan may not live too long. Hardy worked very hard to successfully convince his colleagues in the Royal Society, and so Ramanujan was elected FRS in 1918. Ramanujan was also made Fellow of Trinity College that same year. Although Hardy was saddened to see Ramanujan leave for India in 1919 as a sick man, he said that everyone should be proud that Ramanujan was returning to his homeland in glory and with a reputation that transcended all human jealousies!

Move to Oxford The departure of Ramanujan in 1919 left a gap in Hardy's life. He was also dissatisfied that his presence was not sufficiently appreciated in Cambridge. So he accepted the Savillian Chair of Geometry at Oxford University in December that year. Hardy's close friend and collaborator J.E. Littlewood succeeded him as Cayley Lecturer in Cambridge. As professor at Oxford, Hardy had much more freedom to lecture on topics to suit his tastes and schedule. Since the Savillian Professorship was for geometry, he lectured on that subject in Oxford and gradually added number theory and the theory of functions to his lectures. Hardy humorously remarked "I do not claim to know any geometry, but I do claim to understand quite clearly what geometry is." Hardy's eleven years at Oxford were the happiest of his life. He was taken much more seriously at Oxford than at Cambridge. Colleagues used to look forward to his presence at the afternoon teas and other occasions, where he was the center of attention. While at Oxford, his collaboration with Littlewood continued unabated.

Return to Cambridge In 1931 Hardy accepted the Sadlerian Chair at Cambridge which for many years was occupied by the great mathematician Arthur Cayley, and also briefly by his former professor Augustus Love. At Cambridge, Hardy was allowed to stay in the rooms at Trinity College, whereas there were no such similar facilities at Oxford. By the time Hardy returned to Cambridge, Littlewood was appointed to the Rouse Ball Professorship there. Although Hardy had created a school in analysis at Oxford that was second to none, Cambridge was the world's centre for mathematics, and so it was only appropriate that he returned to spend the rest of his life.

Honours for Hardy The 1920s were years of honours for both Hardy and Littlewood. Hardy was awarded the Royal Medal of the Royal Society in 1920. He served as Secretary of the London Mathematical Society from 1917 to 1926 and never missed a single meeting of the Society. He was President of the Society during 1926–1928 and received the Society's highest honour, the De Morgan Medal, in 1929. In 1926 Hardy founded the Journal of the London Mathematical Society and subsequently a new series of the Oxford Quarterly Journal of Mathematics. Ironically, Hardy died on 1 December 1947, the very day he was to be presented the Copley Medal of The Royal Society.

Eccentricities, Habits and Beliefs Hardy was a bachelor all his life and was wedded to mathematics. He was an avid tennis player and loved to watch cricket. He was a sworn atheist, and indeed dismissed the Goddess of Namakkal's influence on Ramanujan as mere fable. He would never wear a watch or use a fountain pen and hated to have his picture taken. On one occasion he gave the following list of New Year resolutions: (1) Prove the Riemann Hypothesis, (2) make 211 not out in the fourth innings of the last test match at the Oval, (3) find an argument of the non-existence of God which would convince the general public, (4) be the first man on top of Mt. Everest, (5) be proclaimed the first president of USSR, Great Britain, and Germany, and (6) murder Mussolini.

Estimation of Ramanujan Hardy had the highest admiration for Ramanujan. He said that his contact with Ramanujan was the one romantic incident of his life. He was also felt fortunate that he was the only person who "collaborated with Littlewood and Ramanujan on somewhat equal terms." When invited to speak at Harvard University for their Tercentenary Celebrations, he chose to speak about Ramanujan's work. Hardy's twelve lectures on Ramanujan based on those given at Harvard are models of lucidity in exposition. On one occasion in ranking mathematicians on the basis of pure talent, he gave Ramanujan the highest score of 100. On this scale, he gave the great German mathematician Hilbert a score of 80. Hardy gave himself a score of only 25 but said his colleague Littlewood merited 30 since he was the more talented of the two. During the Ramanujan Centennial in Madras in 1987, Atle Selberg speculated that the great German mathematician Hecke would have been a better mentor for Ramanujan than Hardy. Selberg's observation was based on the fact that many of Ramanujan's most significant discoveries lay in the area of modular forms, where Hecke was an expert, and an area which was not Hardy's cup of tea.

But there are other factors that are crucial in mentoring, such as a common language of communication, willingness to spend time with the pupil, and above all, mutual respect. Hardy did everything he could to encourage Ramanujan. The above rankings summarise best Hardy's modesty and his respect for Ramanujan. Both Hardy and Ramanujan blossomed as a consequence of their interaction. As Hindu's we may believe that the Goddess of Namakkal had a role in this—in a dream she instructed Ramanujan's mother to give permission for her son to sail to England, and Hardy may not subscribe to this! Whether or not the Hardy–Ramanujan contact was a Providential Decree, we all should feel fortunate that these two mathematicians, superb in their own ways, met and produced beautiful mathematics that has made a lasting impression.

Chapter 12
J.E. Littlewood: Ramanujan's Contemporary and Hardy's Collaborator

Littlewood

J.E. Littlewood, an outstanding analyst and number theorist, was one of the most eminent British mathematicians of the twentieth century. He was a contemporary of Ramanujan. In addition to his own fundamental contributions, Littlewood is equally famous for his collaboration with G.H. Hardy, Ramanujan's mentor. Indeed, the Hardy–Littlewood collaboration is considered to be the most remarkable and successful partnership in mathematical history, both in terms of its excellence

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and longevity. The Hardy–Ramanujan collaboration, even though it was brief, was equally significant. Although Littlewood was Ramanujan's contemporary, he did not collaborate with the Indian genius. But owing to closeness with Hardy, and because several mathematical problems were of common interest, there are connections between Littlewood and Ramanujan. In this article which is a natural sequel to mine on Hardy last year, I shall describe the life and contributions of Littlewood, discuss his collaboration with Hardy, and point out connections with Ramanujan. I have relied on the book *Cambridge Scientific Minds* by Harman and Mitton and the article by Robin J. Wilson for biographical details of Littlewood and the honors he received; for connections between Littlewood and the work of Ramanujan, I have utilised Hardy's Twelve Lectures on Ramanujan.

Early Life and Education J.E. Littlewood was born on 9 June 1885 in Rochester in Kent as the eldest son to his parents. His father who was himself a Wrangler in the 1882 Cambridge Mathematical Tripos, turned down a Fellowship at Oxford, and opted to become a headmaster of a new school near Cape Town. Thus Littlewood lived in South Africa from 1892 to 1900. Only after he returned to England in 1900 was Littlewood exposed to good mathematics by the distinguished algebraist F.S. Macauley whose approach was not to spoon feed his students, but to make them work independently. Macauley's dictum was "Try a hard problem. You may not solve it, but you will prove something else." Littlewood later adopted this philosophy when training his own students.

Arrival in Cambridge Littlewood obtained an Entrance Scholarship to Trinity College at Cambridge University where he began residence in October 1903. He received training for the Tripos under R.A. Herman, described by Hardy as "the mildest of the most ferocious of the Huns." Littlewood emerged as the Senior Wrangler in Part I of the Tripos, yet he felt he wasted time gaining expertise in solving exceedingly difficult problems against the clock. In Part II of the Tripos, he was placed in Class 1, Division 1.

After a vacation in 1906, Littlewood began research at Cambridge under the direction of E.W. Barnes who gave him a hard problem on integral functions of zero order. Luckily, Littlewood cracked this one and came out with a substantial paper. Next, Barnes asked Littlewood to work on the Riemann Hypothesis, the most celebrated problem in analytic number theory! Littlewood, who liked challenges, took this up and managed to obtain some worthwhile results which he wrote up as a dissertation for a Trinity Junior Fellowship that he received, but only in the following year. Meanwhile, he was awarded the Smith's Prize and received a lectureship at the University of Manchester which he accepted. After going to Manchester, he realised that the work load there was too heavy, and so he returned to Cambridge in 1910 as College Lecturer at Trinity, succeeding Alfred North Whitehead whose lectures had inspired him as an undergraduate. In the very next year Littlewood proved a profound converse of a famous theorem of Norwegian mathematician Abel on the summation of series. Several years later, Littlewood reflected on this achievement and said "On looking back, this seems to me to mark my arrival at a reasonably assured judgement and taste, and the end of my education."

Collaboration with Hardy It was soon after he returned to Cambridge that Littlewood began his collaboration with Hardy which lasted about 35 years until Hardy's death in 1947. As is often the case with successful collaborations, the personalities and styles of Hardy and Littlewood were very different. Rather than being similar, what was important here was that their working methods and ideas complemented each other's perfectly to provide a broader scope and impact. Littlewood was considered the more imaginative of the two, was immensely powerful in analytical methods, and enjoyed challenging problems. Hardy was the finest mathematical craftsmen, had an eye for beautiful mathematical structure, and a superb writing style. In writing a joint paper, Littlewood would provide a logical skeleton in shorthand, and Hardy always wrote the final draft.

To ensure that their partnership would not infringe on their independence and freedom, and to maintain their friendship and mutual respect, they formulated the following four axioms for their collaboration:

(1) When one wrote to the other, it should not matter whether what was written was right or wrong.
(2) When one received a letter from the other, he was under no obligation whatsoever to read it, let alone answer it.
(3) Although it did not matter if they both thought about the same detail, still, it was preferable that they should not do so.
(4) It did not matter at all if one of them had not contributed the least bit to the contents of a paper under their common name.

The great Danish mathematician Harald Bohr (brother of the Nobel Laureate physicist Niels Bohr) remarked "Seldom—or never—was such an important and harmonious collaboration founded on such apparently negative axioms."

In the post-Newtonian era, although there were eminent mathematicians in England, British mathematics took a back seat compared to far greater strides that were made in mainland Europe. Hardy and Littlewood, who dominated the scene in England in the first half of the twentieth century, were attempting to resurrect the glory of British mathematics. And the situation changed because Hardy and Littlewood created a school of analysis unequalled throughout the world. As one English colleague observed, "Nowadays there are only three really great English mathematicians: Hardy, Littlewood, and Hardy–Littlewood!"

Connection with Ramanujan Naturally, as someone who was so close to Hardy and as one working on problems at the interface between Number Theory and Analysis, there are strong links between Littlewood and Ramanujan. When Ramanujan's letters arrived from India, the baffled Hardy consulted Littlewood for an estimation. Indeed it was Littlewood who told Hardy that Ramanujan is in the class of Euler and Jacobi. Also, in connection with that famous episode of the taxi-cab number 1729, it was Littlewood who remarked that "every number is a personal friend of Ramanujan," because Ramanujan was able to provide the most interesting property of any given number.

In 1913, Bertrand Russell wrote a letter to a friend saying "I found Hardy and Littlewood in a state of wild excitement because they believe they have found a second Newton, a Hindu clerk in Madras making 20 pounds a year."

After Hardy and Ramanujan wrote their path-breaking paper in 1917 in which they applied the Farey dissection of the unit circle to obtain an asymptotic formula for the number of partitions of an integer, Hardy and Littlewood over the next few decades developed this "circle method" into a powerful tool of wide applicability encompassing various additive questions in number theory. They wrote a series of influential papers under the title *Some problems in partitio numerorum*. Vaughan's 1997 Cambridge Tract gives an excellent treatment of the Hardy–Littlewood method.

In discussing the validity of various astonishing claims made by Ramanujan, Hardy pointed out that in the study of prime numbers, Ramanujan (and other leading mathematicians) were sometimes misled by intuition. For example, in connection with the distribution of prime numbers, there were many who believed that $\pi(x)$, the number of prime numbers not exceeding x, was smaller than $\ell i(x)$, called the logarithmic integral of x, which is the integral of the reciprocal of $\log(x)$ from 2 to x. All numerical evidence pointed to this. It was Littlewood who showed that in fact the difference $\pi(x) - \ell i(x)$ changes sign infinitely often. Littlewood's method did not yield the first value of x when the sign change would take place. Subsequently the British mathematician Skewes showed that the first such sign change would take place definitely before

$$10^{10^{10^{34}}}.$$

Hardy humorously remarked that this must the largest number to have ever served a definite purpose in mathematics. He also notes that this number is much larger than the number of protons in the universe, which is of the order of 10^{80}.

During the First World War, Littlewood became a Second Lieutenant in the Royal Garrison Artillery and therefore was away from Cambridge. Thus he did not collaborate with Ramanujan. During this period Littlewood improved methods for calculating trajectories of anti-aircraft missiles. Hardy, who on the other hand was as ardent pacifist, remained in Cambridge and therefore could work with Ramanujan. Although he did not collaborate with Ramanujan, Littlewood was instrumental in working with Hardy to get Ramanujan elected as Fellow of the Royal Society (FRS).

Honors and Recognitions In a long and distinguished career, Littlewood received various coveted honors and recognitions in a steady stream. Being younger than Hardy who was considered more senior and prominent, recognitions for Littlewood followed those for Hardy. Littlewood succeeded Hardy as the Cayley Lecturer in Cambridge. Littlewood was elected Fellow of the Royal Society in 1916. He received the Royal Medal of the Society in 1929, nine years after Hardy. In 1928 Littlewood was given the newly formed Rouse Ball Professorship in Mathematics at Cambridge University, a very appropriate appointment because he had been one of Rouse Ball's favourite pupils in Trinity. That same year Littlewood received his first Honorary Doctorate, from the University of Liverpool.

Littlewood succeeded Hardy as President of the London Mathematical Society during 1941–1943. Like Hardy he received the De Morgan Medal of the Society in 1938. Littlewood was also awarded the Sylvester Medal in 1943 and the Copley Medal in 1953.

Littlewood retired in 1950 at the statutory age of 65. He suffered for many years from a nervous disorder which in 1960 was cured by a brilliant neurologist by the discovery of a new drug. This gave Littlewood a new lease of life, and at the age of 75 he started visiting the United States accepting lecture engagements. He continued his mathematical researches well into his eighties. He died on 7 September 1977 at the age of 92.

Hardy's Two Great Partnerships G.H. Hardy remarked that no other mathematician can boast having had the privilege of collaborating with Littlewood and Ramanujan on something like equal terms. Whereas the Hardy–Littlewood collaboration achieved so much over an extended period, the Hardy–Ramanujan partnership produced spectacular results in a remarkably short span of time. On the Hardy–Ramanujan asymptotic formula for partitions Littlewood commented "We owe the theorem to a singularly happy collaboration of two men, of quite unlike gifts, in which each contributed the best, most characteristic, and most fortunate work that was in him." In ranking mathematicians on the basis of pure talent, Hardy gave Ramanujan the perfect score of 100, the great German mathematician Hilbert the score of 80, himself the score of only 25, and Littlewood the score of 30. These ratings of mathematicians reflect Hardy's humility, his respect for Littlewood as the more brilliant of the two, and his highest admiration for the genius of Ramanujan.

Chapter 13
Niels Henrik Abel: Norwegian Mathematical Genius

Abel

Niels Henrik Abel, the great Norwegian mathematical genius, lived only for twenty seven years. Like Ramanujan, Abel fought poverty and other hardships, and in his brief life made path-breaking contributions to several branches of mathematics. Abel and Jacobi, simultaneously and independently, discovered the elliptic functions while considering the inversion of certain integrals. Abel was the first to establish rigorously that there is no general formula to solve all polynomial equations

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of degree five; subsequently Galois established the impossibility of the solvability of polynomials of degree at least five, and this led to the creation of group theory. In the course of his research Abel realised the need for a thorough treatment of infinite series. He not only obtained several important results on convergence of series, but also provided a rigorous basis for mathematical analysis, an area that was shaped subsequently into a solid mathematical theory by the French mathematician Cauchy. Abel's contributions place him among the greatest mathematicians in history. In 2002, on the occasion of the bicentennial of his birth, the Norwegian Academy of Science and Letters, established the Abel Prize for Mathematics, matching the Nobel Prize in prestige and prize money. Here I shall describe the life and contributions of Abel and make comparisons with the life and work of Ramanujan. I will also describe the purpose of the Abel Prize and what led to its creation. For facts about Abel's life, I have relied on accounts by E.T. Bell and by O'Connor and Robertson.

Early Life Niels Henrik Abel was born on 5 August 1802 as the second child of Soren Georg Abel and Anne Marie Simonsen. Soren was a vicar. The family was living at the Finnoy vicarage in western Norway. Later the family moved to Gjerstad in Southern Norway where Abel grew up with five siblings. Although Sorenan was outstanding parishioner, he lived during a period of severe economic crisis in Norway and thus was unable to provide the financial support his family needed. There were various political conflicts at that time involving England, France, Denmark, Sweden and Norway, as a consequence of which the continental powers blockaded England which countered by blockading Norway. The twin blockade was disastrous to Norway which relied on timber exports to England and grain imports from Denmark, and this precipitated the economic crisis. Thus Abel's life was dominated by poverty. Throughout history, we see examples of geniuses who have made outstanding discoveries in spite of financial difficulties that plagued their lives. To such gifted minds like Abel and Ramanujan, the pursuit of knowledge was the primary goal, and they did not let such hardships stand in the way of their intellectual progress.

Exposure to Mathematics In 1815, at the age of 13, Abel was sent to the Cathedral School in Christiana (now Oslo). Although this was a good school, it was in a bad state during the time Abel joined. In this uninspiring atmosphere, Abel initially did not show any talent for mathematics. Educational reforms in Norway in the early nineteenth century replaced the traditional arrangement of one teacher dealing with all subjects for a certain class, by teachers who taught only certain subjects. Unfortunately, the school's mathematics teacher was very old fashioned in his teaching methods; he would insist on students copying the material from the blackboard and box the ears of any pupil who did not understand the subject. In one instance, he punished a pupil so severely, that the poor student died! The mathematics teacher was immediately dismissed, and a new mathematics teacher Bernt Holmboe joined the school in 1817. This was a welcome change for Abel, because Holmboe began

to give individual projects to his students. Holmboe soon discovered the immense talent in sixteen year old Abel and started tutoring him privately. Thus within a year of Holmboe's arrival, Abel was reading the works of Newton, Euler, Lagrange and Laplace. Unfortunately, tragedy struck Abel's family at this time; Abel's father died in 1820. Thus there was no money to support Abel in school, and on top of this Abel had to shoulder the responsibility of supporting his mother and the family.

Fortunately, with Holmboe's help, Abel was able to obtain a scholarship to complete his school education and in 1821 joined the University of Christiana which was founded just ten years before. The university offered studies in theology, medicine and law, but nothing at that time in the natural sciences. Thus Abel pursued mathematics on his own by borrowing books from the university library. Thus he could pursue his investigations on the quintic equation, a study that he had begun during his final year at school.

Unsolvability of the General Quintic In high school we are taught the quadratic formula which provides the roots of any quadratic equation in terms of radicals involving the coefficients of the equation. We are told that there are similar formulae for roots of cubic and quartic polynomial equations, but these are more complicated, and so, in school we are not taught these formula. It turns out that there is no such general formula that will apply to all quintic equations. Many mathematicians prior to Abel had unsuccessfully tried to resolve this problem, but it was Abel who finally proved the unsolvability of the general quintic. Actually, in 1821 Abel first announced his result on quintics in a paper submitted to the Danish mathematician Ferdinand Degen for publication in the Royal Society of Copenhagen. When Degen asked Abel to supply a numerical example in support of the method, Abel found an error in his argument. Abel later fixed the mistake in his proof and in 1823 published a few papers on the quintic and related topics in Norway's first mathematics journal *Magazin for Naturvidenskaberne*.

Although Abel had visited Copenhagen in the interim period, some of the professors who were providing support for Abel realised that he needed to visit the great centres in Paris and Göttingen and come in contact with eminent mathematicians like Gauss and Cauchy. For this reason, he received a government grant but was deemed too immature to travel abroad rightaway. In preparation for his trip abroad, Abel arranged at his own expense the publication of a short paper devoted just to the quintic. The problem, as Abel would realise later, was that the paper was too condensed and so it did not attract the attention it deserved. Subsequently he published an expanded version on this topic about which he said "Mathematicians have been very much absorbed with finding the general solution of algebraic equations, and several of them have tried to prove the impossibility of it. However, if I am not mistaken, they have not as yet succeeded. I therefore dare hope that mathematicians will receive this memoir with good will, for its purpose to fill this gap in the theory of algebraic equations."

Ramanujan and Radicals In the conventional school education in India, we are asked to do some problems involving surds (radicals). Ramanujan was a master at identities involving radicals and made some startling evaluations. There is a charming article on Ramanujan and radicals by Bruce Berndt, H.H. Chan, and L.C. Zhang that appeared about seven years ago in which some of Ramanujan's identities involving radicals are discussed. It is to be noted that Ramanujan was interested in studying equations that can be solved and finding algorithms and relations involving the solutions. He was not interested in proving that certain equations did not have a formula for solutions, or did not have solutions of a certain type. Thus, the unsolvability of the quintic and higher-degree polynomials, or Fermat's Last Theorem, did not attract Ramanujan's attention.

Discovery of Elliptic Functions Abel sent his six-page paper on the quintic to several mathematicians including Gauss whom he intended to visit. He obtained a scholarship in 1825 for the purpose of visiting Göttingen and Paris. Abel arrived first at Copenhagen and was distressed to find that Degen had died. In Copenhagen he was given a letter of introduction to Leopold von Crelle of Berlin who had plans to launch a new periodical *Journal für die Reine und Angewandte Mathematik* (Journal for the Pure and Applied Mathematics). Abel was expecting a reply from Gauss, but that never came. Two reasons for Gauss' silence are given. The first is that Gauss had himself proved the impossibility of solving the quintic and was willing to let Abel take the credit. The second was that Gauss was not interested in solvability of equations by radicals, since he had expressed in his 1801 thesis that such an approach is no better than devising a symbol for the solution and declaring that the equation has a root equal to that symbol! In any case, Gauss' silence disappointed Abel, who changed his plans and went to Berlin instead to meet Crelle.

Abel's contact with Crelle was one of the most fortuitous events of his life. Abel at that time was studying certain algebraic integrals and had a profound realization that one really ought to investigate their inverse functions. These turned out to be doubly periodic functions and are called elliptic functions. The integrals themselves are called elliptic integrals, because the expression of the arc length of an ellipse is one such integral. Jacobi, simultaneously and independently, had come to the same realisation, and so Abel and Jacobi are the founders of the theory of elliptic functions. Abel wrote a series of fundamental papers on elliptic functions and integrals which were published in the first few issues of Crelle's journal. In fact, the very first issue of Crelle's journal, which appeared at the dawn of 1826, contained seven papers of Abel! Thanks in a large part to Abel's papers, Crelle's journal became very soon one of Europe's leading journals. Indeed, Crelle's journal is one of the top mathematics journals even today and is perhaps the oldest mathematics journal among all those that are now in publication.

Ramanujan and Elliptic Functions In studying the works of Ramanujan, one is always led to the following question: How much did Ramanujan know about a particular topic and how much of it did he create himself? Ramanujan obtained several marvellous results on elliptic functions, and so the question of whether he had

access to books in India on this topic is of interest. As Hardy said, Ramanujan never claimed to have invented elliptic functions. Hardy felt that Ramanujan must have learnt the basics of elliptic function theory from books at libraries in Madras, such as the one due to Greenhill. However, Bruce Berndt of the University of Illinois, who has thoroughly analysed Ramanujan's notebooks and edited them in five volumes, holds the following view: Although there is evidence that Ramanujan was acquainted with the books of Greenhill and Cayley, his treatment of elliptic functions was so different that indeed his development was entirely his own. Of course the person who could have resolved this issue was Hardy because Ramanujan was meeting him on a regular basis at Cambridge University. During discussions Hardy could have asked for the source of Ramanujan's ideas, but he did not. Hardy regrets not having asked such a question, but then says that at every meeting Ramanujan was presenting a dozen new identities. So Hardy did not ask Ramanujan how he got them because he spent all the time trying to understand these wonderful claims and prove them!

Infinite Series In the course on his investigations on the quintics, Abel was led to discuss the convergence of the binomial series. Abel realised the need to have a rigorous treatment of infinite series in general and proceeded to provide a systematic treatment. After Abel, it was Cauchy who was the principal force in developing a theory to understand infinite series as we know today. Abel's name is associated with infinite series in many ways. There is the well-known Abel summation formula. Also, there are results which are generally called *Abelian theorems*. An Abelian theorem is, roughly, one which asserts that, if a sequence or function behaves regularly, then some average of the sequence or function behaves regularly. A method of summation is called regular if it sums every convergent series to its ordinary sum. The interest in summability methods is that they provide a way to understand series which are not convergent. There are, for instance, important summability methods due to Abel, Euler, Cesaro, and others. Generally, any theorem asserting the regularity of a summability method is called an Abelian theorem. The direct converse of an Abelian theorem is usually false, because if the regularity theorem for a summation method is reversible, then the method would be trivial, since it applies only to convergent series. It is therefore important to obtain corrected forms of false converses to Abelian theorems, by attaching supplementary criteria. Such criteria are called Tauberian conditions, and the conditionally converse results, Tauberian theorems, named after the mathematician A. Tauber who first proved such a result.

Ramanujan's Theory of Infinite Series Ramanujan was a master in the manipulation and transformation of infinite processes, be they infinite series, continued fractions, integrals, or radicals. Sometimes Ramanujan did these operations formally, without rigorous proofs. Fortunately, he rarely made mistakes. He had his own *theory of constants*, whereby he would attach a value called *the constant of a series* to an infinite sum. Ramanujan's theory is related to the Euler–Maclaurin summation formula. One of his most startling claims is that the infinite sum $1+2+3+\cdots$ of the positive integers equals $-1/12$. Ramanujan would show

this result to his friends and teachers, who all said that this outrageous claim was definitely wrong and that he should study the theory of infinite series before toying with them. We now realise that this claim of Ramanujan is very significant, because the series can be thought of formally as the representation of the Riemann zeta function at the point -1, but then the value of the Riemann zeta function at -1 is $-1/12$! This fact follows from a certain functional equation of the Riemann zeta function after the process of analytic continuation. It is amazing that Ramanujan obtained the correct value without a real understanding of complex variable theory and analytic continuation.

In connection with the rigorous treatment of infinite series, Abel wrote, "Divergent series are the invention of the devil, and it is shameful to base on them any demonstration." Fortunately, now, we understand infinite series very well, and are therefore able to appreciate the nature and importance of the work of Abel, Ramanujan, and others, from a proper perspective. Hardy's book on Divergent Series has a discussion both on Abelian theorems and Ramanujan's method.

The Paris Treatise Although Abel published substantially in Crelle's Journal, he saved what he believed to be the best for the Paris Academy. Upon reaching Paris from Berlin, he worked on what would be called the Paris Treatise that he submitted to the Academy in October 1826. In this memoir, Abel obtained among other things, an important *addition theorem* for algebraic integrals. It is also in this treatise that we see the first occurrence of the concept of the *genus* of an algebraic function. Cauchy and Legendre were appointed referees of this memoir. In Paris, Abel was disappointed to find little interest in his work, which he had saved for the Academy. He wrote to Holmboe "I showed the treatise to Mr. Cauchy, but he scarcely deigned to glance at it." Unfortunately, Abel's memoir was mislaid at the Paris Academy, and so Abel was convinced the work was lost forever. The disappointment of being ignored by both Gauss and Cauchy hit Abel hard. In this miserable state he encountered tuberculosis and therefore decided to return to Norway. Crelle offered him an editorship of his journal, but Abel declined it. Those who provided the scholarship for Abel considered his trip a failure because he did not connect with either Gauss or Cauchy. So Abel's grant was not renewed, and therefore he had to take a loan that he was never able to repay.

Around about that time, Jacobi came up with a paper on the transformation of elliptic integrals which was sent to Abel in 1828. Abel quickly demonstrated that Jacobi's results were consequences of his own. This motivated Abel to write several more papers, but his health was declining rapidly. He died on 6 April 1829.

Posthumous Recognition Ironically, two days after Abel's death, it was announced that Abel's Paris Treatise had been found, and it was praised as an outstanding work. Legendre saw the ideas of Abel and Jacobi and said "Through these works you two will be placed in the class of the foremost analysts of our times." The following year, the Paris Academy's Prize was posthumously awarded to Abel, with the money going to his mother in Gjerstad, Norway. Unaware of Abel's death, Crelle wrote to him saying that he finally succeeded in obtaining a permanent post

for him in Berlin. Crelle said “As far as the future is concerned, you may now rest easy. You belong among us and will be secure. You will be coming to a good country, to a better climate, closer to science, and to sincere friends who appreciate you and are fond of you.”

Recognition for Ramanujan Although Ramanujan had his share of hardships in life, he was more fortunate than Abel in the sense that he was recognised for his contributions during his life time. When he wrote to Hardy, the British mathematician responded favourably. Ramanujan said in his second letter to Hardy “I find a friend in you who views my labours sympathetically.” Concerned that Ramanujan may not live much longer, Hardy moved heaven and earth to get Ramanujan elected Fellow of the Royal Society (FRS) and Fellow of Trinity College in 1918.

The Abel Prize In 1902, for the Abel Centenary, three tasks were specified. The first was to arrange a cultural commemoration. The second was to build a worthy monument in memory of the genius. The third was to establish an international prize. While the first two tasks were completed in 1902, the plans for the Abel Prize were set aside. The Nobel Prize awarded by Sweden is generally acknowledged as the most prestigious prize in the sciences. Alfred Nobel announced his plans for the prize towards the end of the nineteenth century. However, there is no Nobel Prize in mathematics, and this actually prompted Sophus Lie, a world-reknowned Norwegian mathematician, to lobby for a prize in the name of Abel for mathematics. Unfortunately, Lie died in 1899, and so the effort to create the Abel Prize did not materialise.

The most prestigious prize for mathematics has been the Fields Medal awarded at the International Congress of Mathematicians which is held once every four years. Two to four Fields medals are awarded at each of these congresses to mathematicians under the age of forty. The Fields Medal carries only a cash prize of 1,500 Canadian dollars and so does not match the Nobel Prize in prize money. On 23 August 2001, a year prior to the Abel Bi-Centenary, it was announced that the Norwegian Government would establish an Abel Fund worth NOK 200 million. Part of the fund’s money would be used by the Norwegian Academy of Science and Letters to award an annual *Abel Prize* for mathematics worth about $750,000, which is of the order of magnitude of the Nobel Prize, and a long awaited one. Unlike the Fields Medal, the Abel Prize is for lifelong contributions to mathematics. Jacob Palais, President of the International Mathematical Union, said in 2002 that the Abel Prize will alter the global mathematical landscape and raise the visibility of mathematics in society. The first Abel Prize was awarded in March 2002 to Jean-Pierre Serre of France. Serre had won the Fields Medal in 1954 as a young mathematician. The second Abel Prize awarded in March 2004 was shared by Michael Atiyah of Oxford University and I.M. Singer of MIT. Atiyah had won the Fields Medal in 1966.

The Ramanujan Prize Just this year, the Abdus Salam International Centre for Theoretical Physics (ICTP) announced the creation of a *Ramanujan Prize* of $10,000 to be awarded annually to a mathematician under 45 of the Third World.

When Abdus Salam created the ICTP in the early 1960s, it was with the intention of strengthening science in the Third World. Thus, it is not surprising that the Ramanujan Prize created by the ICTP is for mathematicians from Third World countries. Interestingly, the Ramanujan Prize of the ICTP is also supported by the Abel Foundation.

A Game of Youth Hardy has stressed that the greatest mathematicians made their most significant discoveries when they were very young. He pointed out that Galois who died at 20, Abel at 29, and Riemann at 40 had actually made their mark in history. So Hardy said that the real tragedy of Ramanujan was not his early death at the age of 32, but that in his most formative years, he did not receive proper training, and so a significant part of his work was rediscovery. It is with this same view that it has been argued that with the age restriction for Fields Medals, one is not missing out much in recognising outstanding mathematicians. But the Abel Prize will recognise mathematicians of all ages.

Leopold von Crelle said that Abel is one of those rare beings that nature produces barely once a century. Such high praise applies equally well to Ramanujan. There is much for us to learn from the lives of Abel and Ramanujan. In particular, we should aim to follow their example and not let anything stand in the way of lofty intellectual pursuit.

Chapter 14
Issai Schur: Ramanujan's German Contemporary

Schur

Issai Schur (1875–1941), the great German mathematician, made fundamental contributions to many branches of mathematics, most notably to algebra. In algebra, Schur was a pioneer in the study of groups and their representations. In addition to algebra, Schur's research spanned number theory, divergent series, integral equations and function theory. In his research in algebra, Schur studied a variety of combinatorial problems. It was this that led Schur to independently discover and

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prove the Rogers–Ramanujan identities. Indeed, by studying these identities combinatorially, Schur discovered in 1926 an important result now known as Schur's partition theorem, which is some ways is the next level result beyond the Rogers–Ramanujan identities. Schur's partition theorem has since become one the most fundamental results from which have emerged various hierarchies of partition identities. In this article I shall first describe briefly the life and contributions of Schur. I will then discuss the connections between the work of Schur and Ramanujan by focussing on Schur's proof of the Rogers–Ramanujan identities and the implications of this which include Schur's partition theorem and important current research on related topics. Although Schur is most known for his contributions to group theory, here I have chosen to highlight his research on partitions because of connections with the work of Ramanujan. For biographical details, I have relied on an article by O'Connor and Robertson. For insights regarding Schur's work on the Rogers–Ramanujan identities, I have profited from discussions with Professor George Andrews.

Early Life and Contributions to Group Theory Issai Schur was born on 10 January 1875 in Mogilyov in Russia, which is located in what is now known as Belarus. Although he grew up in Russia, he spoke German without an accent. This was something that was helpful to him during much of his professional career in Germany during the Nazi regime when persons of non-Aryan descent were persecuted. At the age of 13, he moved to Latvia where he attended the Gymnasium in Libau, now called Liepaja.

At the age of 19, Schur was admitted to the University of Berlin for higher studies in mathematics and physics. There he was strongly influenced by Frobenius, who was one of his teachers, and under whom he did his doctoral dissertation. Frobenius and Burnside were the principal founders of the representation theory of groups as groups of matrices. Group theory which has its origins in the work of Galois around 1830, underwent significant development in the hands of Cauchy, Jordan, Cayley and others, in the ensuing decades. Representation theory became a powerful tool in the study of groups, and Schur was fortunate to learn it from one of its founders. As a consequence, Schur made important contributions to the theory of group representations by himself and in collaboration with Frobenius. In his doctoral thesis of 1901, Schur investigated the rational representations of the general linear group over complex numbers. Certain functions that Schur introduced in this thesis are called S-functions. Subsequently, they have been called Schur functions and have applications to the representation theory of the symmetric group as well as to the study of plane partitions. Thus interest in Schur's thesis continues even today.

During the period 1904–1907, Schur worked on the projective representations of groups and in that process discovered a very fundamental result that is known today as Schur's Lemma. Also, in a series of papers he introduced and developed a concept now known as Schur multipliers. Samuel Eilenberg and Saunders Maclane, the founders of Category Theory, noticed in 1949 that some of their

ideas had been anticipated by Schur decades earlier in the study of Schur multipliers.

Professional Life in Berlin Schur became a lecturer first at the University of Berlin in 1903 and held this position until 1911 when he moved to Bonn where he was offered a professorship. However, he returned to Berlin in 1916 and was promoted to full professorship there in 1919. He held this position with distinction until his tenure was ended by the Nazis in 1935. During this period he built a powerful school with several active students and assistants. This school was the most influential mathematical group in Berlin and among the most prominent in all of Germany. As the leader of the group, Schur inspired all those around him. Several students who did their doctorates under him became leading mathematicians: Richard Brauer and his brother Alfred, Richard Rado, Bernhard Neumann and Walter Ledermann, to name a few. Ledermann describes Schur as a teacher par-excellence: "Schur was a superb lecturer. His lectures were meticulously prepared … I remember attending his algebra course in a lecture theatre filled with 400 students. Sometimes when I had to be content with a seat at the back of the lecture theatre, I used a pair of opera glasses to get at least a glimpse of the speaker."

In developing this school, Schur also founded the journal *Mathematische Zeitschrift*, which is one of the leading journals today published by Springer. In recognition of his contributions, Schur was elected to the Prussian Academy in 1922 at the suggestion of the great physicist Max Planck, who was Secretary of the Academy.

The Rogers–Ramanujan Identities In 1914, the focus of Schur's research shifted from algebra to other areas, most notably to certain combinatorial questions. In particular, the theory of partitions founded by Euler caught his attention.

One of the first results an entrant to the theory of partitions encounters is Euler's theorem which states that *the number of partitions of an integer into distinct parts equals the number of partitions of that integer into odd parts.* In 1917 Schur independently discovered the two Rogers–Ramanujan partition theorems of which, for simplicity, we state only the first, namely, *the number of partitions of an integer into parts that differ by at least* 2 *equals the number of partitions of that integer into parts of the form* $5k+1$ *or* $5k-1$.

The Rogers–Ramanujan partition theorem could be viewed as the next level result beyond Euler's theorem (but is much deeper!) if we restate Euler's theorem in the following fashion: *The number of partitions of an integer into parts that differ by at least* 1 *equals the number of partitions of that integer into parts of the form* $4k+1$ *or* $4k-1$. Thus by increasing the minimal permissible difference from 1 to 2 on the one hand, integers of the form $4k+1$ and $4k-1$ are replaced by those of the form $5k+1$ and $5k-1$.

Neither Rogers nor Ramanujan emphasised the partition-theoretic aspect of the two identities they had discovered independently. Ramanujan discovered the two

series = *product* identities in India prior to his departure to England in the course of studying a certain continued fraction that arises by taking the ratio of the two series. This continued fraction has fascinating transformation properties which Ramanujan used to make some tantalising evaluations that he communicated to Hardy in his two letters of 1913. Thus this continued fraction is of importance in the theory of modular forms.

Ramanujan did not have a proof of these identities. Neither Hardy nor his colleague Littlewood at Cambridge University could supply a proof. Hardy's friend MacMahon, who at that time was preparing a book on Combinatory Analysis, formulated the identities in partition-theoretic form but stated these as unproved conjectures in his book. It was only in 1917 while going though some old issues of the Proceedings of the London Mathematical Society that Ramanujan noticed a trilogy of papers by the British mathematician L.J. Rogers who, two decades earlier, had proved these and other identities. Rogers had proved them purely by q-series techniques and introduced some important ideas that became the basis of the work of Bailey and Slater to discover many more such identities. Thus Ramanujan was motivated by transformation properties, and Rogers by q-series techniques, in the discovery of these fascinating identities. But neither Rogers nor Ramanujan stated the combinatorial form of the identities that Schur did. Also, Rogers and Ramanujan were unaware of the work of Schur, and vice versa, because there was no communication between England and Germany at that time owing to World War I.

Just as the approaches of Rogers and Ramanujan had important consequences, the combinatorial view of Schur also led to major advances some of which we describe next.

Schur's Partition Theorem and Implications By emphasising the partition aspect of the Rogers–Ramanujan identities, Schur in 1926 discovered the next level result which is now known as Schur's partition theorem: *The number of partitions of an integer into parts that differ by at least* 3*, and with no consecutive multiples of* 3 *as parts, equals the number of partitions of that integer into parts of the form* $6k+1$ *or* $6k-1$. Thus Schur noticed that in addition to the condition that parts must differ at least 3, one needs an *extra condition*, namely, that consecutive multiples of 3 should not occur as parts.

Schur's partition theorem has been of central importance in the theory of partitions and has been the starting point of several fruitful lines of investigation.

In 1967–1968, George Andrews discovered two infinite hierarchies of partition theorems of the Rogers–Ramanujan type emanating from Schur's theorem. More precisely, Schur's theorem is the first case of each of these two infinite hierarchies, and this is also the only instance where the hierarchies coincide. Later in 1971, in a computer aided project funded by the IBM, Andrews discovered a new companion result to Schur's partition theorem.

More recently, in 1993, Basil Gordon and I proved a substantial refinement and generalization of Schur's partition theorem by a new technique called *the method of weighted words.* This led to new connections between Schur's theorem and multinomial coefficients and the discovery of companions to Schur's theorem that eluded

the computer search of Andrews. Subsequently, I have developed the method of weighted words in collaboration with Andrews, Gordon and Alexander Berkovich, to obtain results in a different direction from that taken by Andrews in 1967–1968, namely, the study of a deep partition theorem of Göllnitz and its extension.

Schur's proof of the Rogers–Ramanujan identities has also been a basis for the important bijective proof of the Rogers–Ramanujan identities in 1980 due to Adriano Garsia and Steve Milne. Often in combinatorics when an identity establishes the equality to two sets defined in different ways, one desires a *bijection*, namely, a one-to-one correspondence that converts members of one set to the other in a natural fashion. Such a bijective proof of the Rogers–Ramanujan identities was found by Garsia and Milne by formulating an *involution principle* whose basic ideas can be traced to Schur's original combinatorial proof of these identities. Thus Schur's proof of the Rogers–Ramanujan identities and his 1926 partition theorem have had major impact in the theory of partitions.

Schur also made other contributions to partitions. In a joint paper with K. Knopp he discusses elementary ways to estimate $p(n)$, the total number of partitions of n. This precedes the elementary derivation of the leading term in the asymptotic formula for $p(n)$ due to the great Hungarian mathematician Paul Erdös in 1942 that appeared in The Annals of Mathematics. As is well known, Hardy and Ramanujan in a path-breaking paper gave an exact formula for $p(n)$ in the form of an asymptotic series by introducing a powerful technique called the *circle method.* In the introduction of this magnum opus, Hardy and Ramanujan discuss estimates for $p(n)$ by elementary means.

A Sad End Although Schur had a very successful tenure for nearly two decades in Berlin, Nazi persecution eventually took its toll. On 7 April 1933, under a law passed by the Nazis, persons of non-Aryan descent could not hold certain appointments and had to be retired. When Schur's classes were cancelled, there was an outcry among students all of whom admired and respected him. The well-known German mathematician Bieberbach (after whom a famous conjecture on univalent functions is named) joined the Nazi opposition against Schur who was dismissed from his position in Berlin in 1935. Yet, Schur continued to live in Germany and work on mathematics from his home. He had invitations to the United States and Britain which he declined. Like Ramanujan who pursued mathematics with a passion even from his deathbed, Schur continued his fundamental research from home even though he was stripped of his position. In 1938 Bieberbach strongly objected to Schur continuing to be a member of the Prussian Academy, following which Schur was forced to resign from the Academy. Unable to face this persecution any longer, Schur left Germany for Palestine in 1939. But he was a broken man by then. The final humiliation was that he had to find a sponsor just to pay the *Reich's flight tax* that would permit him to leave Germany. Since he did not have sufficient funds, he decided to sell his books to the Institute for Advanced Study in Princeton where Einstein and other brilliant scientists had joined. Schur died in Tel Aviv on his 66th birthday.

Geniuses in Their Own Way Schur and Ramanujan were very different as mathematicians, both in training and approach to research. Ramanujan did not have formal collegiate education, whereas Schur had solid training under Frobenius. Schur's forte was algebra and combinatorial understanding, whereas Ramanujan's work was in the realm of analysis, interfacing with number theory. Although Schur and Ramanujan took entirely different approaches in the study of partitions, they both discovered, independently of each other, what are now called the Rogers–Ramanujan identities. The approaches of both Schur and Ramanujan continue to have a major influence and will provide food for thought for mathematical researchers in the decades ahead.

Chapter 15
Robert Rankin: Scottish Link with Ramanujan

Rankin

Robert A. Rankin, the eminent Scottish mathematician, died on 27 January 2001 at the age of 86. He was a central figure in the world of Ramanujan for several reasons: (i) he made fundamental research contributions to many important problems considered by Ramanujan, (ii) he worked closely with Hardy, Littlewood, and Watson, the three principal British mathematicians who were associated with Ramanujan, and (iii) he published historical articles and books on Ramanujan's life. Rankin's seminal work on Ramanujan's tau function and coefficients of cusp forms was the starting point of some major developments in the theory of modular forms. His two books *Ramanujan: Letters and Commentary* and *Ramanujan: Essays and*

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Surveys, coauthored with Bruce Berndt, have attracted the attention of mathematics enthusiasts, students, and laymen around the world, and so the international community is now much more aware of Ramanujan's life and mathematical contributions. Thus he made a broad impact in the world of Ramanujan. In this article I will provide a brief outline of the life and accomplishments of Rankin, explain some of his important contributions that relate to Ramanujan's work, and describe his two visits to India in the 1960s and during the Ramanujan Centennial. In preparing this article I have relied on the biography of Rankin by O'Connor and Robertson and the description of the life and work of Rankin by Bruce Berndt, Ken Ono, and Winfred Kohnen in the Special Issue of the Ramanujan Journal in Rankin's memory.

Boyhood Years and Schooling Robert Rankin was born on 27 October 1915 in Garlieston, in Wigtownshire, Scotland. His parents Reverend Oliver Shaw Rankin and Olivia Theresa Shaw Rankin were first cousins. Oliver Rankin was a parish minister of Sorbie, Wigtownshire, when Robert was born. Robert was named after his paternal grandfather who was a pastor. Rankin went to Garlieston School and at an early age developed a keen interest in Gaelic culture and language, an interest that continued throughout his life. Subsequently he went to Fettes College in Edinburgh. In December 1933, at the age of eighteen, he travelled from Edinburgh to Cambridge University to take a scholarship examination. Although he did not do as well as he had hoped in that examination, he was awarded a minor scholarship at Clare College in Cambridge which he accepted in 1934.

The Cambridge Years At Cambridge, Rankin was strongly influenced by the lectures of the eminent number theorists Littlewood and Ingham. He attended their lectures while studying for Part III of the Tripos Exam after having successfully emerged as Wrangler in Part II. He completed his BA degree in 1937. He then took research in number theory under the supervision of Ingham, who described Rankin as the most serious among all his gifted pupils. Rankin obtained very fundamental results on differences between consecutive prime numbers and was awarded the Raleigh Prize in 1939. For his accomplishments, he was honoured by being elected Fellow of Clare College. This work in Cambridge subsequently led to four more important papers by him on this topic in the next decade about which there will be more in the sequel.

Under Hardy's Influence From Cambridge Rankin became a research assistant of G.H. Hardy, who was at the University College in London at that time. Since Hardy was deeply involved in analysing the results of Ramanujan, Rankin was also drawn to Ramanujan's work under Hardy's influence. In his doctoral dissertation Rankin obtained fundamental results on Ramanujan's tau function. His three papers on this topic which appeared in the *Proceedings of the Cambridge Philosophical Society* between 1939 and 1940 were perhaps his most influential works.

Word War II When World War II came, Rankin wanted to join the British Army as a soldier, but it was decided that his talents would be put to better use at the Ministry of Supply in Kent. There Rankin began his work on the trajectory of rockets. He was later transferred to North Wales where he continued his investigation of rocket trajectories until the end of World War II. This work was classified until the end of the war, but in 1949 he published it as a massive paper in the *Philosophical Transactions of the Royal Society*, the longest paper ever to be published in that journal! While in North Wales he met his future wife Mary Llewellyn who was working as a secretary at the Ministry of Supply.

Back to Cambridge When the war was over, Rankin returned to Cambridge University in 1945 as Faculty Assistant Lecturer at Clare College. After serving as Assistant Tutor, he was promoted to the rank of Lecturer in 1948. At this time he also served as Secretary of the Cambridge Philosophical Society and Editor of its Proceedings. One of his colleagues at Cambridge had the following to say of him: "He was a conscientious teacher and had a wide interest in mathematics. Those who took the trouble to ask serious questions were rewarded with precise and serious answers."

Rankin also served as Director of Studies in Mathematics at Clare. He performed this task efficiently but felt the business of supervision as very time consuming. In spite of these heavy duties, Rankin enjoyed being in Cambridge. He left Cambridge in 1951 to go to Birmingham to take up the post of Mason Professor of Mathematics which was vacated by G.N. Watson. His stay in Birmingham was brief, because in 1954, he accepted the Chair of Mathematics at the University of Glasgow, in his home country Scotland, a position he held with distinction until his retirement.

Illustrious Career in Glasgow For 28 years at the University of Glasgow, Rankin made important research contributions and rendered outstanding service to the university and the profession. He wrote over 100 research papers in number theory, modular forms, and function theory. His two books, *The modular group and its subgroups* (1969), and *Modular forms and functions* (1977), were published during this period. After serving as Dean of the Faculty of Science at Glasgow, he was Clerk of the Senate from 1971–1978 and Dean of Faculties during 1986–1988. And throughout his tenure at Glasgow from 1954 to 1982, he was Head of the Department of Mathematics. These administrative duties did not prevent him from making signal contributions to research. Indeed he managed both research and administration admirably.

Rankin's productivity continued even after his retirement. His two books with Bruce Berndt *Ramanujan: Letters and Commentary* and *Ramanujan: Essays and Surveys* were published in 1995 and 2001. I will now briefly discuss some significant mathematical contributions of Rankin and emphasise connections with Ramanujan's work appropriately.

Gaps Between Prime Numbers The distribution of prime numbers remains an active topic of investigation to this day. In particular, the study of gaps between

consecutive prime numbers continues to pose challenges to mathematicians. As a consequence of the Prime Number Theorem, it is well known that if p_n is the nth prime number, then the gap (difference) $p_{n+1} - p_n$ has average size about $\log n$, the logarithm being taken to the base e. Two natural questions immediately arise: (i) Can the ratio $(p_{n+1} - p_n)/\log n$ be arbitrarily large? Equivalently, can this ratio tend to infinity over a sequence of numbers? (ii) How small can this ratio be?

Although very simple to formulate, both are difficult questions. The celebrated prime twins conjecture, which remains unsolved, asserts that there are infinitely many pairs of primes differing by 2. The gap 2 would be the smallest possible to occur infinitely often, but we are far from proving this fact. Paul Erdös, one of the most eminent mathematicians of the twentieth century, proved that there is an infinite sequence of values of n for which this ratio is less than a certain constant ℓ which is less than 1. Ten years later, in a paper that appeared in the Proceedings of the American Mathematical Society, Rankin showed that ℓ is less than 0.977. That it took a decade to effectively determine an upper bound for ℓ shows the difficulty of question (ii).

With regard to the first question, Rankin showed that over an infinite sequence of integers n, the ratio goes to infinity at least as fast as a certain function given in terms of iterated logarithms of n. Lower bounds like this are quite difficult to obtain. There is a famous \$10,000 problem of Erdös concerning an improvement of this lower bound of Rankin, which is unsolved.

In the course of proving these results on gaps between primes, Rankin studied the number of integers below a given magnitude x that have all their prime factors below a specified value y. He obtained upper bounds for the number of such integers by a very ingenious and simple technique, now known as *Rankin's method* which is widely applicable.

Ramanujan's Tau Function The tau function of Ramanujan, which arises as the coefficient in the expansion of a certain infinite product, is one of the most fundamental in the theory of modular forms. In a seminal paper published in 1939 in the *Proceedings of the Cambridge Philosophical Society*, Rankin obtained an upper bound for the tau function, and more generally, upper bounds for coefficients of certain modular forms called cusp forms. Ramanujan had conjectured a specific upper bound for the tau function (*The Ramanujan Hypothesis*), and the bound Rankin obtained was not far away from it. Ramanujan's conjecture for the tau function was solved only in the 1970s by Pierre Deligne using deep methods from algebraic geometry. Atle Selberg, one of the greatest mathematicians of the twentieth century (who was Professor at the Institute for Advanced Study in Princeton), extended Rankin's method in the 1960s in a significant way. This technique is now called the *Rankin–Selberg* method. Selberg predicted that the resolution of the Ramanujan Hypothesis would require techniques from algebraic geometry, and this turned out be the case with Deligne's proof. Deligne received the Fields Medal in 1978 for this work and is now at the Institute for Advanced Study.

Ramanujan's Lost Notebook After Ramanujan died, the loose sheets of paper on which Ramanujan had jotted down formulas while on his death bed were collected by his widow Janaki. Subsequently they were sent to Hardy in Cambridge. These sheets contained several formulae on mock theta functions. This manuscript of Ramanujan eventually came to Watson's possession because, as an authority in the area of special functions, he was best suited to analyse these formulae. Watson did analyse some of the formulae and delivered a talk on third-order mock theta functions as his retiring Presidential Address to the London Mathematical Society. Watson continued to investigate the mock theta functions, but no one else but him had access to these loose sheets.

Rankin knew G.N. Watson quite well, especially from his stay at Birmingham. (Rankin succeeded Watson at Birmingham as Mason Professor of Pure Mathematics when the latter retired.) When Watson died, his widow requested Rankin to collect the papers and manuscripts in Watson's study and have them documented and filed suitably. Rankin did collect these and with the help of J.M. Whittaker, Watson's famous collaborator in the study of special functions, placed them in the Wren Library at Trinity College, Cambridge, as desired by Watson. These loose sheets of Ramanujan that were in Watson's possession all along were in the set of papers and manuscripts filed by Rankin in the Wren Library. Strangely, the mathematical world forgot about the location of this manuscript of Ramanujan. They were found quite accidentally by George Andrews of The Pennsylvania State University in 1976 while he was looking through some papers in the Watson collection at the Wren Library. These loose sheets of Ramanujan were given the name The Lost Notebook. The story of their rediscovery by Andrews is part of mathematical folklore.

Two Historical Books on Ramanujan In collaboration with Bruce Berndt of the University of Illinois, Rankin wrote two historical books on Ramanujan. The first of these entitled *Ramanujan: Letters and Commentary* was published by the American and London Mathematical Societies in 1995. In this book, all letters to, by, and about Ramanujan are collected, and detailed commentaries made about each letter. If there is a mathematical formula in a letter, then this formula is analysed, and appropriate references to papers in the literature are given. If there is a statement about a mathematical practice or tradition, then there is a description as to how that evolved or how it is implemented. For example, in one of the letters there is a reference by Hardy about Ramanujan's election as Fellow of the Royal Society. Therefore in this book there is a commentary on what such an election means and how it is done. If a letter contains a reference to a food item from India, then there is a description as to how it fits into Indian cuisine. Thus this is a book that appeals to mathematicians, enthusiasts of mathematics, and laymen.

The second book, *Ramanujan: Essays and Surveys*, contains a magnificent collection of articles about Ramanujan and those close to him. There is an article by Selberg in which he describes how Ramanujan was a strong influence on him. There is also an article by Jonathan and Peter Borwein on *Ramanujan and pi*.

The impact of these two books has been enormous because of their wide appeal. I had the pleasure of reviewing both books for the American Mathematical Monthly (see articles 21 and 22 in this book) .

Two Visits to India Rankin visited India twice. In his first visit in 1968, he gave a course of lectures at the Ramanujan Institute in Madras. These lectures formed the basis for his book *The modular group and its subgroups* published in 1969. Rankin also published a paper on Ramanujan's tau function in the Symposia on Theoretical Physics and Mathematics of MATSCIENCE, Institute of Mathematical Sciences, Madras, at the invitation of my father Professor Alladi Ramakrishnan, who was then Director of MATSCIENCE. Rankin's second visit to India was in 1987 for the Ramanujan Centennial. In addition to speaking at the conference in Madras, he attended a few more conferences in India and one in Sri Lanka arranged in connection with the Ramanujan Centennial.

Honours and Recognitions Rankin received several honours and recognitions in his long and distinguished life. He was elected Fellow of the Edinburgh Mathematical Society in 1955 and received the Keith Prize of that Society. The London Mathematical Society awarded him the Senior Whitehead Prize in 1987 and the De Morgan Medal in 1998, the Society's most prestigious honour.

The Ramanujan Journal When I requested Rankin to join the Editorial Board of the Ramanujan Journal at its inception in 1997, he graciously agreed. Thus he was a Founding Editor. The presence of a person of his eminence on the Editorial Board, and his close association with the mathematics of Ramanujan, contributed to establishing this journal on a firm footing and raising its stature. After Rankin died, a Special Issue of The Ramanujan Journal was brought out in his memory. For those of us who knew him, it was a privilege to have had his association, not only because of his mathematical eminence, scholarship, character, and dignity, but also because he brought us closer to Ramanujan.

Chapter 16
Ramanujan and π

The number π, the ratio of the circumference of a circle to its diameter, has fascinated mathematicians through the centuries and continues to intrigue researchers even today. Giants like Archimedes, Newton, Euler and Gauss have through their seminal work contributed substantially to our current understanding of the properties of π and shown how this has a direct bearing on many fundamental questions. And to the impressive list of luminaries who have studied this number, Ramanujan's name must be added. As is usually the case when Ramanujan confronts a topic, he provides a touch of magic to it by means of his incredibly beautiful formulae. In a famous paper published in 1914 in The Quarterly Journal of Mathematics (Oxford) entitled "Modular equations and approximations to π", Ramanujan has several tantalising formulae involving π and other numbers, and such expressions are the basis for recent computations of the digits of π to over two billion decimal places! This paper contains work done by Ramanujan in India prior to his departure to England. It is amazing that Ramanujan, who in rural India wrote many of these formulae on a piece of slate and erased them with his elbow, should remain alive in the modern world of the computer! In this article I will discuss in lay terms some of the important properties of π and how we understand these in relation to many fundamental problems. In doing so, I will describe some of Ramanujan's observations on π and how they influence current research. In preparing this article, I profited greatly from the book entitled "Pi and the AGM" by Jonathan Borwein and Peter Borwein as well as from comments by Bruce Berndt.

Early History The realisation that the ratio of the circumference of a circle to its diameter is the same for all circles, is an important landmark in human history. This invariant number is denoted by the Greek letter π (pi). One finds in the Egyptian Rhind Papyrus, which dates about 2000 BC, the approximate value $(16/9)^2 = 3.1604\ldots$ for π. With regard to the creation of the earth, implicit in The

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Bible is the statement that π is nearly equal to 3. In attempting to compute the circumference of the earth, the ancients were motivated to calculate π to a high degree of accuracy. Eratosthenes of Alexandria, who is remembered mainly for the *Sieve*, a procedure to generate prime numbers, actually computed the circumference of the earth. But the one figure from that era who towers above every one is Archimedes of Syracuse (287–212 BC). Indeed, Archimedes is considered to be one of the five greatest scientific thinkers of all time along with Newton, Euler, Gauss and Einstein. Archimedes was a master of approximation and of the limit process, and in the course of computing the areas and volumes of various geometrical figures, he even anticipated Newton and Leibniz in the development of integral calculus. By computing the lengths of the inscribed and circumscribed polygon of 96 sides for a circle of unit radius, Archimedes showed that π was less than $3\frac{1}{7}$ and greater than $3\frac{10}{71}$. We realise today that these early calculations of Archimedes are indeed the first few steps in the harmonic–geometric mean iteration which can be programmed in the computer to give remarkably good approximations to π.

Squaring the Circle The members of the Pythagorean school believed that all phenomena could be expressed in terms of integers (whole numbers) and rationals (ratios of integers). For those steeped in this philosophy, it must have come as shock when the square root of 2 was shown to be irrational (not rational). The number π also resisted all attempts to produce an exact rational value, increasing the suspicion that it too might be an irrational number. But the irrationality of π was proved only in 1761.

Struggling to understand π geometrically, the Greeks posed the problem of squaring the circle. More precisely, the problem was to construct using only the ruler and compass, a square which is equal in area to a given circle. Since the area of a circle of unit radius is π, what was required was the construction of the side of the corresponding square which will be the square root of π units in length. This is one of the *three problems of antiquity*. The second problem is to trisect any given angle using only the ruler and compass. (It is easy to bisect any given angle, and indeed this construction is taught in the early years of high school.) The third problem is to double the cube, that is construct a cube, again with only ruler and compass, which is twice the volume of a given cube. This is equivalent to asking for the construction of the cube root of 2 using ruler and compass. The geometrical construction of the square root of 2 is easy, because this is the hypotenuse of a right-angled isosceles triangle whose equal sides are of unit length, a fact known to any high school student of Euclidean Geometry. All three problems of antiquity are now known to be impossible because of the pioneering work of Galois in the 19th century, whose study of the solutions of algebraic equations laid the foundations of Group Theory. From the work of Galois it followed that the only numbers which could be constructed using ruler and compass are special types of algebraic numbers. (Algebraic numbers are those which arise as solutions of polynomial equations with integer coefficients.) Both the square root and the cube root of 2 are algebraic numbers, but the cube root of 2 is not of the special type. In the case of the problem of squaring the circle, its impossibility is a consequence of the fact that π is not an

algebraic number. This was proved by Lindemann in 1882 and is considered to be one of the crowning achievements of 19th century.

Ramanujan was very much interested in the problem of squaring the circle. In a note published in the Journal of the Indian Mathematical Society (1913), he offered a geometrical construction to obtain an approximation for the square root of π based on the observation that π is nearly $355/113$. Ramanujan discusses this approximation again in his famous paper of 1914. The most commonly used rational approximation to π is $22/7$, and we understand this now in terms of the continued fraction for π. The number 3 is the first approximation that emerges from the continued fraction, with $22/7$ as the second (and better) approximation. The number $355/113$ considered by Ramanujan is the third approximation. Each successive approximation from the continued fraction is better than the preceding one. Even before the 15th century, the Chinese and Indian mathematicians were aware that $355/113$ was an extremely good approximation to π.

Representations for π The invention of Calculus paved the way for our present understanding of π and other numbers. Between 1665 and 1666, Newton himself calculated π to about 15 decimal places by means of an infinite series for the arcsine function. His contemporary and rival in mainland Europe, Leibniz, produced in 1674 a more elegant expression for π using the inverse of the tangent function, a fact that was observed independently by the Scottish mathematician James Gregory in 1671. Indeed the Gregory series formed the basis for the calculation of π to 71 decimal places by the British astronomer Edmund Halley and his student Abraham Sharp. Other infinite expressions for π were provided during this period. One of the most beautiful and well-known expressions is an infinite product due to John Wallis involving the even numbers in the numerator and the odd numbers in the denominator. In fact Wallis challenged Lord Brouncker by saying "I bet you can't top this!" Lord Brouncker, who was the first president of The Royal Society, was not a mathematician. Nevertheless, he accepted a challenge and produced a lovely continued fraction expansion for π. Since Lord Brouncker did not give a proof of his derivation, it remained a mystery as to how he arrived at his result. Bruce Berndt points out that Ramanujan has several fascinating continued fractions in his notebooks. One of these continued fraction formulas of Ramanujan for a ratio of gamma functions yields Lord Brouncker's fraction as a special case by setting the variable $x = 1$. Interestingly, Ramanujan had communicated this continued fraction along with many other results in his first letter to Hardy in 1913.

Although these infinite expressions for π due to Wallis and his contemporaries were important in understanding many fundamental problems, no one at that time was able to use these formulae to prove that π is irrational. Leonard Euler (1707–1783), the most prolific mathematician in history, was the supreme master in the manipulation of infinite expressions. He produced what is perhaps considered to be the most beautiful and important formula in all of mathematics connecting e, the natural base of the logarithms, i, the imaginary square root of minus 1, and π. Euler's formula is that e to the power $i\pi$ equals minus 1. It is said of Euler that he could calculate with as much ease as a fish takes to water or an eagle takes to the

wing! (I think the same could be said of Ramanujan.) Using his superior powers of calculation, Euler evaluated several infinite series and products in terms of π.

The first proof of the irrationality of π was supplied by Lambert in 1761. Legendre subsequently improved on this and showed that the square of π is irrational, and expressed the opinion that π may not even be an algebraic number, a belief that was shared by Euler. Transcendental numbers, that is numbers which are not algebraic, were not even known to exist at that time. The first transcendental numbers were constructed by Liouville only in 1840. Then in 1873, Charles Hermite showed that e, the natural base of the logarithms, is transcendental. Finally, in 1882, Lindemann, extending the ideas of Hermite, and using Euler's formula, established that π is a transcendental number and thus settled the 2300 year old problem of squaring the circle.

Elliptic and Theta Functions There are many who feel that the greatest mathematical discovery of the 19th century is that of the elliptic and theta functions due primarily to Abel, Jacobi and Weierstrass, each working independently of the others.

Most of us are familiar with the trigonometric functions, sine, cosine, tangent etc. We know that the values of these trigonometric functions repeat, that is they are periodic functions with period 2π. In the study of calculus, it was observed that the values of certain integrals involving special quadratic polynomials lead to inverses of the trigonometric functions. But even mild variations of these polynomials lead to integrals which are very difficult to evaluate. Such integrals arise for example in the study of the circumference of an ellipse, whose computation is obviously of interest because the planets move around the sun in elliptical orbits. The major realisation was that the inverses of certain of these integrals lead to functions which have two periods, one a real number and another a complex number. These are the elliptic functions. The connection between elliptic and theta functions is due to fact that special combinations of theta functions in the form of ratios yield elliptic functions.

Elliptic and theta functions have become very important because of their wide applicability ranging from Statistical Mechanics to Number Theory. The modern notion of an Elliptic Curve in Algebraic Geometry is a far reaching extension of the basic idea of an elliptic function and one that has proved to be extremely important. It is the Theory of Elliptic Curves blended with Number Theory that led to Andrew Wiles' recent proof of Fermat's Last Theorem. Elliptic curves are used today in the fastest algorithms to factor large numbers and to test whether a given large number is a prime number. And elliptic functions and the relations they satisfy, especially some observed by Ramanujan, are the crucial tools employed in the present day calculations of the digits of π.

The AGM The arithmetic mean (average) of two positive numbers is one half of their sum, while their geometric mean is the square root of their product. It is easy to see that the arithmetic and geometric means lie between the two numbers. It is interesting to note that the geometric mean is always less than the arithmetic mean. Exploiting this simple idea, Gauss produced the remarkable arithmetic–geometric mean iteration. More precisely, Gauss starts with two positive numbers a and b with

b less than a. Let c and d be their arithmetic and geometric means respectively. Then c and d lie between a and b with d being smaller than c. Gauss then calculates the arithmetic and geometric means of c and d to get numbers which are even closer and repeats this procedure indefinitely. An infinite sequence of pairs is thus generated whose difference keeps shrinking, and so these numbers have a limit. This limit is the arithmetic–geometric mean (AGM) of a and b and is denoted by $M(a, b)$. For any number a larger than 1, Gauss evaluated the AGM of a and 1 to be an elliptic integral involving π and the trigonometric function sine, and thus established a connection with the theory of elliptic functions. He then expressed the opinion that this connection would open up a whole new field of analysis.

In the past decade, the AGM and other means obtained by iterative processes have been studied extensively because of their close connection with the elliptic and theta functions and also because these iteration procedures give rapid methods to calculate π and other related numbers. Jonathan Borwein and Peter Borwein have studied such convergence questions for a large class of numbers and especially for π, and it was their work which led D.H. Bailey to compute several million digits of π. More recently, Kanada in Japan has employed the iteration techniques of the Borwein's and computed 1.6 billion decimal digits of π.

Ramanujan Hardy expressed the opinion that Ramanujan did not have a grasp of complex variable theory and that this was the cause for some of the slips that Ramanujan made in the theory of prime numbers. What is most baffling is that Ramanujan had a complete mastery over elliptic and theta functions, a subject in which significant contributions cannot really be made without a firm grasp of complex variable theory. Ramanujan's theory of elliptic functions was work that he did in India prior to his departure to England. Hardy was of the belief that Ramanujan did not invent elliptic functions by himself, that he must have had access in India to Greenhill's book or other books on these topics from which he must have learnt some of the basic ideas. In any case, Ramanujan's work on elliptic and theta functions was work that he did before he was exposed to sophisticated techniques by Hardy. Of Ramanujan's remarkable ability to evaluate elliptic and other definite integrals, Hardy has said that during the course of his lectures, if at any time he needed the value of a certain integral, he would simply turn towards Ramanujan in the audience, who would provide the answer instantly!

Ramanujan published an important paper in the Oxford Quarterly Journal of Mathematics (1914) entitled "Modular equations and approximations to π". This paper contains a myriad of formulae including transformation formulas for elliptic and theta functions called modular relations. In fact Ramanujan has discovered more modular relations than Abel, Jacobi and other luminaries combined! Ramanujan's approach to elliptic and theta functions is so original and his notation so different from that of his illustrious predecessors, that contrary to Hardy's opinion, one is inclined to believe that Ramanujan discovered these results without prior knowledge of the subject. Ramanujan's approach to this theory is now gaining acceptance as can be seen from recent lectures by Bruce Berndt entitled "Ramanujan's theory of theta functions".

In addition to modular identities, this paper contains several series representations for the reciprocal of π and for numbers which are of the form π divided by the square root of an integer. Since these series converge very rapidly, it was realised that they could be used to calculate π and other numbers to a high degree of precision. During the past decade, William Gosper used one of Ramanujan's series for the reciprocal of π to evaluate 17 million terms in the continued fraction expansion of π. Within the last two years, the brothers David and Gregory Chudnovsky utilised certain extensions of some of Ramanujan's formulae to compute π to about two billion decimal places. The most outstanding thing about their calculation was that the Chudnovsky brothers did this by assembling a computer (by mail order) in their own apartment in New York, a computer built specifically for this purpose!

Why Calculate the Digits of π Many may wonder what is achieved by calculating millions of digits of π. Is it simply for the challenge? Of his calculation of π to 15 decimal places, Newton admitted "I am ashamed to tell you to how many figures I carried these computations, having no other business at this time." Sir Edmund Hillary's response when asked why he chose to climb Mount Everest was "because it is there!"

In the case of the calculation of the digits of π, there is more at stake than just the challenge. Every attempt to understand π has produced new techniques which have proved applicable elsewhere. The methods developed for studying π shed light on the properties of other numbers. That is also what we hope might happen with regard to the study of *normal numbers*.

A normal number to the base ten is one in whose decimal expansion, every digit from zero to nine occurs with equal frequency, and more generally, any given block of, say, k digits occurs with frequency ten to the power minus k. The number $0.1234567891011121314\ldots$, whose decimal digits are simply obtained from writing down all the positive integers, is an example of a normal number to base ten. We know that *almost all* numbers are normal, but it is extremely difficult to prove that a given number is normal. For instance, we suspect that π and e are normal numbers, but these questions are at present unresolved.

Another problem is to obtain what is called an *irrationality measure* for π, that is study the degree of approximation of π by rational numbers. In 1958, K.F. Roth of Imperial College, London, was awarded the Fields Medal (the equivalent of the Nobel Prize in mathematics) for showing that all algebraic irrational numbers have irrationality measure equal to 2. We know that *almost all* numbers have irrationality measure 2 including most of the transcendental numbers, but given a specific transcendental number, it is usually very difficult to confirm that its irrationality measure equals 2. In the case of π, it is conjectured that the irrationality measure is 2, but we are far away from this result. A few years ago, the Chudnovsky brothers showed that the irrationality measure for π was less than 16.53. They obtained such superior irrationality measures for π and related numbers by employing Ramanujan's formulae in his famous paper of 1914.

Ramanujan's Ability Ramanujan's mastery of infinite processes and his superior powers of manipulation are only too well known. It is always fascinating to find out

what motivated Ramanujan to write a particular formula down. For instance, in his 1914 paper he offers the fourth root of the number $97\frac{1}{2} - \frac{1}{11}$ as an approximation to π and provides a geometrical construction. This approximation may also be found in his second and third notebooks, but no indication is given as to what led him to this result. One possible explanation (due to N.D. Mermin) is that the decimal expansion of the fourth power of π is $97.409091034002\ldots$, and Ramanujan probably observed that the digits 09 appear in succession. So he might have replaced this by the decimal expansion $97.4090909\ldots$ with the digits 09 repeating indefinitely and thus was led to $97\frac{1}{2} - \frac{1}{11}$ which he wrote in a different form. But this raises the question as to what led Ramanujan to consider the decimal expansion of the fourth power of π in the first place. Bruce Berndt has explained this as follows: "Ramanujan's facility with continued fractions was unequalled in mathematical history. As suspected by Mermin, Ramanujan might have known that the continued fraction for the fourth power of π starts as $97\frac{1}{2}$ and very soon has the large integer 16539 in the sixth step of the expansion (sixth partial quotient). Hence he might have concluded that the fourth power of π should have a very good rational approximation and this probably led him to the decimal expansion."

In summary, trying to understand π is as much of a challenge and pleasure as attempting to understand the mind of Ramanujan. It is only fitting that, Ramanujan, the most romantic mathematical figure in history, should have been charmed by π whose undying beauty has captivated mathematicians from the days of Archimedes to the present!

Chapter 17
Ramanujan and Partitions

Although the theory of partitions was founded by Euler two centuries ago, it is no exaggeration to say that it was Ramanujan's spectacular contributions to this field at the beginning of this century that propelled it to a glorious position that it deserves and continues to occupy today. Ramanujan's work on partitions provided many unexpected and important connections with other areas such as number theory, combinatorics, analysis, computer algebra, and physics, and so this subject is now being studied from several points of view. In the past few decades, Professor George Andrews of The Pennsylvania State University has systematically studied a wide class of problems in the theory of partitions and written an Encyclopedia on Partitions, the standard reference in this field. Professor Andrews is thus the torch bearer of the subject today, and to him also goes the credit of our present understanding of many of Ramanujan's identities in the context of partitions. Indeed, to many current researchers like me in this field, Ramanujan is the inspiration, and Andrews the medium to understand Ramanujan's work and its ramifications. In this article I shall discuss some of Ramanujan's most significant contributions to partitions and the progress in several directions it led to. I will also indicate how Ramanujan's work will form the basis for future research. In preparing this article, I have primarily depended on Hardy's classic twelve lectures on Ramanujan, Andrews' Encyclopedia and his CBMS Lectures on partitions, q-series, and allied fields, as well as several stimulating conversations and collaborations with Professors George Andrews and Basil Gordon (University of California, Los Angeles), in addition to my own research in this area.

Partitions By a partition of a positive integer we mean a representation of that integer as a sum of positive integers. Two partitions are considered the same if they differ only in the order of their parts. For example, $3+2+1+1$ and $3+1+2+1$ are the same partition of 7. Denote by $p(n)$ the number of (unrestricted) partitions of n.

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For example, $p(5) = 7$ because there are 7 partitions of 5, namely, 5, $4+1$, $3+2$, $3+1+1$, $2+2+1$, $2+1+1+1$, and $1+1+1+1+1$. There is a technique called the *method of generating functions* whereby one can translate all the information in a sequence of numbers into a single continuous function from the realm of calculus. Euler noticed that the generating function of $p(n)$ possessed a beautiful product representation. Euler also realised that using generating functions, elegant results on partitions could be proved such as: *The number of partitions of an integer into odd parts equals the number of partitions of that integer into distinct (non-repeating) parts*. This is commonly referred to as Euler's theorem. For example, there are six partitions of 8 into odd parts, namely, $7+1$, $5+3$, $5+1+1+1$, $3+3+1+1$, $3+1+1+1+1+1$, and $1+1+\cdots+1$. There are also six partitions of 8 into distinct parts, namely, 8, $7+1$, $6+2$, $5+3$, $5+2+1$, and $4+3+1$. Euler also obtained a very useful recurrence relation for partitions by means of his famous *pentagonal numbers theorem*. With the foundation laid by Euler, the theory of partitions underwent a glorious transformation with the magic touch of Ramanujan. The Indian genius produced a variety of new results which showed connections with many different areas. These results were also startlingly beautiful. I will now explain a few of Ramanujan's discoveries, and for each, I shall describe the very interesting history, give some idea of current research, and indicate future directions.

The Hardy–Ramanujan Formula In one of his letters to Hardy written in 1913, Ramanujan gave a formula for the coefficients of a certain series expansion of an infinite product which suggested that there ought to be a similar exact formula for the partition function $p(n)$ in terms of continuous functions. Hardy felt that this claim of Ramanujan was too good to be true but was convinced that it was possible to develop an asymptotic formula for $p(n)$ in terms of continuous functions. Asymptotic formulas may be thought of as approximate formulas. Generally they do not give an exactly correct answer, but they are close. By an ingenious and intricate calculation involving the singularities of the generating function of $p(n)$ in the unit circle, Hardy and Ramanujan obtained an asymptotic formula, which when calculated up to a certain number of terms, yielded a value which differed from $p(n)$ by a quantity no more than the reciprocal of the square root of n. Since $p(n)$ is an integer, it is clear that the exact value of $p(n)$ is the nearest integer value to what is given by the series. This was indeed amazing, and Hardy wanted to test the correctness of the result. So he asked his friend Major MacMahon to compute the values of $p(n)$ using Euler's recurrence formula. It turned out that the value $p(200) = 3972999029388$ given by Euler's recurrence, was also given by the Hardy–Ramanujan formula! The series representation of Hardy–Ramanujan is genuinely an asymptotic series in the sense that when summed up to infinity, it diverges. Subsequently Hans Rademacher noticed that by making a very mild but important change, namely, by replacing the exponential functions with hyperbolic functions, the Hardy–Ramanujan asymptotic series could be converted into a series that in fact converges to $p(n)$. Actually, in the 1913 letter to Hardy, Ramanujan used hyperbolic functions to claim an exact formula for a related problem, and so Ramanujan was indeed correct in surmising that a similar exact formula would exist for $p(n)$. Professor Atle Selberg of the Institute for

Advanced Study in Princeton, one of the greatest living mathematicians, said during his address at the Ramanujan Centennial in Madras on 22 December 1987 that the exact formula that Rademacher obtained was actually more natural than the Hardy–Ramanujan formula. Indeed Selberg discovered this exact formula on his own but did not publish it once he found out that Rademacher had done it earlier. An aspect of Ramanujan's discoveries that comes up time and again is the surprising and unbelievable form of the results. The exact formula for $p(n)$ that Ramanujan conjectured was considered unbelievably good by Hardy who settled for less, namely an asymptotic formula, and Ramanujan agreed (according to Selberg) out of respect for his mentor. In any case, the Hardy–Ramanujan asymptotic formula gave rise to very powerful analytic method to evaluate the coefficients of series arising in a wide class of problems in additive number theory. This *circle method*, originally due to Hardy–Ramanujan and subsequently developed by Hardy–Littlewood and others, is one of the most widely applicable methods today and will continue to be a major tool in the future.

Ramanujan Congruences As soon as Ramanujan saw the table of values of the partition function that MacMahon had prepared (in order to check the Hardy–Ramanujan asymptotic formula), he (Ramanujan) wrote down three congruences. Hardy was stunned in disbelief when he saw these claims. The first Ramanujan congruence states that *the number of partitions of an integer of the form* $5n + 4$ *is always a multiple of* 5. The second congruence states that *the number of partitions of* $7n + 5$ *is a multiple of* 7. The third congruence is the statement that *the number of partitions of* $11n + 6$ *is a multiple of* 11. What stunned Hardy was that there are divisibility properties for partitions which are combinatorial objects defined by an additive process.

In the table of partitions that MacMahon had prepared, the values of $p(n)$ are listed in columns of length 5 starting with the value $p(0) = 1$ as shown below:

n	$p(n)$
0	1
1	1
2	2
3	3
4	5
5	7
6	11
7	15
8	22
9	30
…	

Thus the values $p(5n + 4)$ are at the bottom of each column, and the lower right-hand digit in each block is either 0 or 5. So in principle, any one staring at the last entry of each column could have observed Ramanujan's first congruence, but one has to be in search of such a property in order to observe it. MacMahon prepared the

table, and Hardy checked it, but neither of them observed the congruence because they were not looking for such surprising connections! Ramanujan who always had the eye for the unexpected, wrote the congruence down as soon as he saw the table.

The study of such divisibility properties or congruences for partition functions and similar objects has been one of the most fruitful and active areas of research in number theory during this century. What is fascinating is the role of the theory of modular forms (an abstract and beautiful area involving analysis, algebra, and number theory) in the study of such congruences.

Although modular forms have been the main tool used in the study of such congruences, combinatorial explanations are of great interest because partitions are combinatorial objects. In 1944, Freeman Dyson, then a young student in Cambridge University, found such an explanation of Ramanujan's first two congruences using the concept of *rank of a partition*. Dyson published this in the undergraduate mathematics journal of Cambridge called *Eureka* and conjectured that a similar explanation ought to exist for the third congruence by means of a different statistic that he called the *crank* and that had not been found yet. Professor Dyson has humourously remarked that this was perhaps the first instance in mathematics where an object had been named before it was found. Dyson's crank remained elusive for many years, but in 1987, immediately after the Ramanujan Centennial Conference at the University of Illinois, Frank Garvan and George Andrews found the crank at last. Frank Garvan is now my colleague at the University of Florida.

A recent path-breaking work in this field is due to Professor Ken Ono of The Pennsylvania State University and the University of Wisconsin. It had been widely believed that the values of the partition function $p(n)$ behave randomly modulo m except for the Ramanujan congruences. Ramanujan had conjectured that if $m \neq 5, 7, 11$, then in every arithmetic progression there are infinitely many values of the partition function which are not multiples of m. Last year, Ono showed that Ramanujan's conjecture is mostly correct. More precisely, Ono proved that for every prime $m \neq 5, 7, 11$, the proportion of arithmetic progressions for which Ramanujan's conjecture holds is greater than $1 - 10^{-100}$. Although this seems like phenomenal evidence, very recently, Ono has shown that Ramanujan's conjecture is actually false! That is by a deep study involving L-functions and Galois representations, Ono showed that for every $m \geq 13$, there are arithmetic progressions with very large moduli, in which the values $p(n)$ are always multiples of m. Thus the Ramanujan-type congruences although very rare are indeed plentiful. Thus the study of Ramanujan-type congruences not only for the partition function but for coefficients of modular forms will continue to be a very active field of research.

Rogers–Ramanujan Identities In the entire theory of partitions and q-series, the Rogers–Ramanujan identities are unmatched in simplicity elegance and depth. The statement of the first identity is: *The number of partitions of an integer into parts differing by at least* 2 *equals the number of partitions of that integer into parts which when divided by* 5 *leave remainder* 1 *or* 4. The statement of the second identity is similar. For example, there are 6 partitions of 10 into parts differing by at least 2, namely, 10, $9+1$, $8+2$, $7+3$, $6+4$, and $6+3+1$. There are also 6 partitions of

10 into parts of the form $5m+1$ or $5m+4$, namely, $9+1$, $6+4$, $6+1+1+1+1$, $4+4+1+1$, $4+1+\cdots+1$, and $1+\cdots+1$. The analytic form of the identities is the equality of two infinite series and two corresponding infinite products, the infinite series being the generating function of the partitions satisfying difference conditions, while the products are the generating function for partitions satisfying congruence conditions.

Ramanujan communicated the analytic form of the identities in a letter to Hardy in 1913. Neither Hardy nor any of his contemporaries could prove them. When Ramanujan arrived in England in 1914, Hardy asked him for a proof, but Ramanujan could not supply one. The partition version of the identities is not due to Ramanujan, but due to MacMahon who was completing a book on Combinatory Analysis at that time. So in 1915 when MacMahon's book was published, these partition theorems were stated as unsolved combinatorial problems.

In 1917, while going through some old issues of the London Mathematical Society, Ramanujan came across three papers (1894–1896) of the British mathematician L.J. Rogers, who had proved not only these identities but many similar ones. For various reasons, the work of Rogers was largely ignored by his British contemporaries, and Ramanujan's rediscovery of his work brought him due recognition. From then on the name Rogers–Ramanujan identities came to stay.

It must be pointed out that like Ramanujan, Rogers also only stated the analytic form of the identities and not the partition version. The German mathematician Issai Schur, working independently, had also proved the Rogers–Ramanujan identities and realised their combinatorial significance. This helped Schur obtain the next level partition theorem, now known as Schur's partition theorem.

Ramanujan's motivation for the identities came from the study of an infinite continued fraction. The ratio of the two Rogers–Ramanujan series produces this continued fraction, which has a lovely product representation modulo 5 owing to the Rogers–Ramanujan identities. Ramanujan noticed several beautiful transformation formulas for this continued fraction and used these to calculate the fraction at various special values. Ramanujan's continued fraction is one of the most fundamental objects in the theory of modular forms.

Nowadays by a Rogers–Ramanujan-type identity we mean an identity in the form of an infinite series equals an infinite product, where the series is the generating function for partitions satisfying difference conditions, and the product is the generating function for partitions satisfying congruence conditions. Rogers–Ramanujan-type identities have arisen in a variety of settings, ranging from in the study of Lie algebras to problems in statistical mechanics. In the 1980s, the Australian Mathematical Physicist Rodney Baxter first observed that the Rogers–Ramanujan identities arose as the solution of the hard hexagon model in Statistical Mechanics. Subsequently, Andrews and Baxter worked out the complete set of solutions. For this work, Baxter was awarded the Boltzman Medal of the American Physical Society. More recently, Professor Barry McCoy of the Institute of Physics in Stony Brook, in collaboration with Alexander Berkovich, Anne Schilling and Ole Warnaar obtained new extensions of various Rogers–Ramanujan-type identities by a study of models in Conformal Field Theory in Physics. For this and other work, McCoy was awarded the Heineman Prize in mathematical physics last year.

In the last few years, in collaboration with George Andrews and Basil Gordon, I have developed a new technique called *the method of weighted words* which can be applied to a large class of Rogers–Ramanujan-type identities to obtain generalizations and multi-parameter refinements. Subsequently, I also developed a theory of weighted partition identities out of this approach, which provided new relationships between many Rogers–Ramanujan-type functions. Thus the study of Rogers–Ramanujan-type identities from several points of view will be a major area of research in the years to come.

Mock Theta Functions There are many who consider the mock theta functions as among Ramanujan's deepest contributions. They were discovered by Ramanujan in India after his return from England a few months before his death. In his last letter to Hardy, Ramanujan states that he has made a very significant discovery, namely the mock theta functions, and lists several of orders 3 and 5.

Some consider the theta functions as the greatest mathematical discovery of the nineteenth century. Theta functions are extremely interesting because they satisfy transformation properties. The mock theta functions are like the theta functions in the sense that the circle method can be used just as effectively to calculate their coefficients, but they do not satisfy exact transformation formulas. The examples Ramanujan sent to Hardy were well-known theta-type series but with mild changes in sign which made them mock theta functions. The Lost Notebook of Ramanujan contains several identities involving mock theta functions. In the course of proving many of these, Professor Andrews has explained the partition-theoretic significance of several mock theta identities. Recently, Professors Basil Gordon and Richard McIntosh have found some new mock theta functions of orders 8 and 12, and established identities for them. The partition interpretation of these new mock theta identities remains to be explored.

Everlasting Beauty The theory of partitions is just one of the ways in which one can enter the garden of Ramanujan's theorems. Shakespeare said in one of his sonnets "Since brass, nor stone, nor earth, nor boundless sea, but sad mortality oversways their power, how in this rage shall beauty hold a plea, whose action is no stronger than a flower?" Ramanujan's mathematics has an everlasting beauty, and as we move into the new millennium, there is no doubt in my mind that his influence will continue be strong in several areas of research.

Chapter 18
Major Progress on a Problem of Ramanujan

There has been sensational recent progress in the study of universal quadratic forms, a subject which originated in 1910 when Ramanujan wrote down 54 examples of such quadratic forms. These recent developments were described by Professor Manjul Bhargava of Princeton University while delivering the Third Ramanujan Commemoration Lecture at SASTRA University, Kumbakonam, on December 22, 2005.

One of the most elegant results in number theory is a 1770 theorem of Lagrange which states that every positive integer is a sum of four squares. This motivated Ramanujan to investigate those quadratic forms which would represent all positive integers. In his notebooks Ramanujan wrote 54 examples of such quadratic forms. Ramanujan's discovery resulted in a flood of activity in the ensuing decades in the study of universal quadratic forms, namely, quadratic forms representing all positive integers.

In 1993, Conway and Schneeberger announced a startling result that in order to decide whether certain special quadratic forms defined via matrices are universal, one need only check whether these represent the integers 1 to 15. Their proof of this result which was very intricate was never published. Professor Bhargava has found a new and much simpler proof of this result using geometric notions which he presented in his lecture.

Conway had conjectured that in the general case of integer-valued quadratic forms, in order to decide which of those are universal, it suffices to check whether a certain special set of 29 integers up to 290 can be represented. This is a very difficult problem, and Conway said that he did not expect to see a proof in his lifetime! Quite surprisingly, during the summer of 2005, Professor Bhargava and Jonathan Hanke proved Conway's conjecture and following this were able to determine the complete list of all 6436 integer-valued quadratic forms that are universal. In establishing this result, they used a variety of techniques and results due to Ramanujan such as the circle method that Hardy and Ramanujan introduced in the asymptotic study of partitions, and Ramanujan's bounds for the coefficients of certain modular forms of integral weight that were proved by Fields medalist Pierre Deligne.

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Professor Bhargava said that Ramanujan would have been quite pleased not only at the complete resolution of the problem of universal quadratic forms, but also at the methods employed in the proofs. “I am pleased to present these results in Ramanujan’s hometown on Ramanujan’s birthday,” he said.

The Ramanujan Commemoration Lecture by Professor Bhargava was the concluding event for the International Conference on Number Theory and Mathematical Physics held at SASTRA University in Kumbakonam. On the opening day of the conference, he and Professor Kannan Soundararajan (University of Michigan) were each awarded the First SASTRA Ramanujan Prizes of $10,000, for outstanding contributions in areas of mathematics influenced by Ramanujan.

Chapter 19
Genius Whom the Gods Loved—A Review of "Srinivasa Ramanujan: The Lost Notebook and Other Unpublished Papers"

Professor G.H. Hardy of Cambridge University, who was Srinivasa Ramanujan's mentor, once remarked that the real tragedy with Ramanujan was not his early death, but that during his most formative years in India prior to his departure to England, his genius was misdirected owing to a lack of formal training. Hardy pointed out that the best original work in any field is usually done before the age of 30 and that this is especially true of mathematics. So if Ramanujan had lived longer, although he would have produced a good deal more, Hardy argued that the quality of his work would not have been greater.

Several years after Ramanujan's death we have come to realise that Hardy may have well been wrong in this estimation of Ramanujan, for the depth and power of Ramanujan's mathematics actually increased with the years. His work in England after contact with Hardy and exposure to more sophisticated techniques shows a greater amount of variety and power. His discoveries in India after return from England shortly before his death are the beginnings of theories which are grander in design and greater in depth. In the context of more modern methods, we now realise the significance of this last work of his on mock theta functions from the contents of the Lost Notebook. Ramanujan was definitely on the rise, and we can only imagine the heights he would have reached had he lived longer.

Ramanujan has no parallel in the history of mathematics. His life story is astounding to his own countrymen and therefore much more so to people of a totally different cultural background. The story of the Lost Notebook is equally amazing.

During the final few months when Ramanujan was bedridden in Madras, he used to ask his wife Janaki Ammal for a scrap of paper to write down various formulae. Ramanujan wrote a letter to Hardy during this period outlining his discovery of the mock theta functions. Shortly after he died, Janaki Ammal collected these sheets and sent them to Cambridge. Many of these sheets were of thin foolscape paper

This article is a review of the book *Srinivasa Ramanujan: The Lost Notebook and Other Unpublished Papers* (with an introduction by George Andrews), published by Narosa Publishing House, New Delhi. This book review appeared in *The Hindu*, India's national newspaper, on February 20, 1988.

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intermingled with thicker quarto sheets like the ones Hardy liked; so Ramanujan must have taken some of these sheets of paper from Cambridge to India. After Ramanujan's death, B.M. Wilson began the task of editing the original two notebooks that Ramanujan had maintained in India prior to his departure to England. Hardy asked G.N. Watson to sort out this last manuscript of Ramanujan mainly because it was more in line with Watson's investigations. After Watson's death this manuscript along with several of Watson's papers were despatched to the Trinity College Library. Why this manuscript was not separated and classified under Ramanujan's name is unclear. At any rate it was forgotten that this manuscript was among Watson's papers, and so it soon acquired the name "The Lost Notebook," the romantic implications of which seemed to fit well with the atmosphere surrounding Ramanujan. In 1976, George Andrews of the Pennsylvania State University, the world authority on partitions and q-hypergeometric series, was in Cambridge looking through the Watson section. Quite unexpectedly he noticed a manuscript with incredibly beautiful formulae, and one look at them convinced him that these could only be written down by Ramanujan and that this was indeed the Lost Notebook! Andrews has been researching on the contents of the Lost Notebook ever since and has supplied proofs of many of the outstanding formulae contained therein.

The Lost Notebook contains about 650 formulae of which about 60 percent are on q-series and mock theta functions. Theta functions are extremely interesting mathematical objects and equally useful as well, especially in physics. The mock theta functions mimic the theta functions in a certain way but are more general objects. Ramanujan wrote down several fascinating formulae for these functions which have interesting interpretations in the theory of partitions. About 30 percent of the Lost Notebook deals with the important topic of modular equations. Ramanujan demonstrates his unusual genius here by producing remarkable connections between this topic and q-series—this was so unexpected that Andrews was shocked on seeing it for the first time! The remaining 10 percent of the Lost Notebook contains miscellaneous results on integrals, Dirichlet series, and asymptotic formulae.

It is indeed appropriate that during the Ramanujan Centennial celebrations in Madras, the Prime Minister released the first published copy of The Lost Notebook and presented it to George Andrews. Narosa Publishing House has done a wonderful job in bringing out in the form of a hard bound book of over 400 large-sized pages containing a faithful reproduction of Ramanujan's last manuscript. Other unpublished papers of Ramanujan are also included in the book as also some correspondence between Ramanujan, Hardy, Littlewood, Watson, and others. George Andrews has provided an illuminating introduction at such short notice which would interest the expert and non-expert as well. This book will be a proud possession to one and all an inspiration to any young researcher.

Original thinkers are to be measured by the impact of their contributions which would stand the test of time. Ramanujan's mathematics has fascinated the world's best minds all these years. Andrews has remarked that the results in the Lost Notebook are only tips of icebergs of grander theories which underlie them. Ramanujan must have envisioned these theories in his own unique way. The mathematical community will spend at least several decades understanding Ramanujan's contributions

and results in the Lost Notebook. The publication of the Lost Notebook is therefore a major event in the world of mathematics, and we are fortunate to witness this exciting period in Indian mathematical history.

Chapter 20
The Discovery and Rediscovery of Mathematical Genius—A Review of "The Man Who Knew Infinity: A Life of the Genius Ramanujan"

The life story of Srinivasa Ramanujan (1887–1920), the legendary mathematician from India, is astonishing in many ways. Without any formal training he produced results of incredible depth and beauty which challenged some of the finest minds in England during the beginning of this century. What is even more remarkable is that he discovered many of these results while in South India, where the traditional Hindu way of life had changed little over the centuries. Professor G.H. Hardy of Cambridge University was immensely impressed by these discoveries and arranged for Ramanujan to go to England. Although Ramanujan had great difficulties adjusting to the British way of life, he wrote several fundamental papers in Cambridge. But the rigors of life in England during World War I, made worse by his own peculiar habits, led to a rapid decline in his health. This forced him to return to India where he died a year later. Even from his deathbed Ramanujan made startling discoveries on mock theta functions about which I shall comment later. In his all-too-brief life of thirty-two years, he has left numerous results, unique in their beauty, deep and fundamental to be of lasting value, which place him among the greatest mathematicians in history. Hardy published twelve lectures in 1940 explaining many facets of Ramanujan's work; in the past few decades, American mathematicians George Andrews, Richard Askey, and Bruce Berndt have published several books and monographs expanding on Ramanujan's ideas. But this is the first detailed biography of Ramanujan, and the author is successful in bringing out the drama in a most interesting manner.

Ramanujan was born on 22 December 1887 in an orthodox South Indian Brahmin family. His parents who lived in Kumbakonam, a small town in what is now the State of Tamil Nadu, were childless for many years. They prayed to the Goddess Namagiri, the deity in the neighboring town of Namakkal, to bless them with a child. Promptly, Ramanujan's mother became pregnant and gave birth to him in the nearby town of Erode, her mother's place. To his parents Ramanujan was a divine gift, and

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This article is a review of the book *The Man Who Knew Infinity: A Life of the Genius Ramanujan*, by Robert Kanigel, Charles Scribner's, New York, 1991. A slightly modified version of this review appeared in The American Scientist, Vol. 80, pp. 388–389 (1992).

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the Goddess of Namakkal was held in great veneration by his family. Ramanujan grew up in a traditional Hindu environment, learning stories from the great epics and verses from the Hindu holy scriptures.

Ramanujan showed signs of his special mathematical talent early. He kept notebooks where he would regularly jot down his findings. What was most strange was the manner in which he arrived at his results, and this still remains a mystery. Often, he would suddenly get up in the middle of the night and immediately write down identities involving infinite series and products. Those near him have said that Ramanujan used to mention of the Goddess of Namakkal appearing in his dreams and presenting him these incredible formulae. As an agnostic, Hardy has dismissed the story of the Goddess as mere fable. That divine inspiration is often the cause of work of exceptionally high quality is more acceptable to a Hindu than to someone steeped in Western tradition. For instance, Hindu belief is that it was the blessing of Goddess Kali which instantly transformed Kalidasa from a shepherd to a poet par-excellence!

In discussing Ramanujan's early years, the author has given a very interesting account of the lifestyle in South India. However, as a South Indian myself, I do not agree with all his observations. For instance, he is critical of certain private practices of orthodox Hindus, interpreting these as prejudices. What the author does not stress is the tolerance of Hinduism to other beliefs. An orthodox Hindu who adheres to certain practices, generally does not impose his beliefs on others. This was the case with Ramanujan who stuck to his Hindu habits in England, even though people around him viewed things differently.

Ramanujan's excessive pre-occupation with mathematics led to his neglect of other subjects, and he had to drop out of college. With the intention of making him more responsible, Ramanujan's mother got him married in 1909 to Janaki, who was then only a nine year old girl. Child marriages were common in India then, but nowadays they are unlawful. While describing Indian marriage customs, Kanigel also discusses various South Indian traditions and superstitions.

Next, Kanigel describes Ramanujan's efforts in approaching influential people for financial assistance so that he could pursue his research unhindered by the distractions of a job. What Ramanujan really needed was the attention of a leading mathematician. India was a British Colony, and so it was natural for him to write letters to British professors stating his results. And it was Hardy who responded favorably.

G.H. Hardy was, at the beginning of this century, leading the revival of British mathematics which had taken a backseat in the post-Newtonian era. Being well known, he used to receive letters from amateur mathematicians making false claims to the solution of famous problems. So, when a letter from Ramanujan dated Jan. 1913 arrived containing a long list of formulae without any proofs, his first reaction was to ignore the letter as one written by a fraud. However, a closer look showed that there were several beautiful formulae some of which defeated him completely. He came to the conclusion that it was more probable for Ramanujan to be a genius, because a fraud would not have the imagination to invent such identities! A correspondence followed, and Hardy invited Ramanujan to Cambridge so that his raw, untutored genius could be given a sense of direction.

Ramanujan's reaction to this invitation and his mother's were negative. Orthodox Hindus believed then that it was a sin to cross the seas. (This is no longer the belief; nowadays, there is a large migration of talented people from India to the United States.) Once again the Goddess of Namakkal provided the solution! This time his mother had a dream where she saw Ramanujan being honored in an assembly of European mathematicians and the Goddess instructed her not to stand in the way of her son's recognition. So, finally, Ramanujan sailed for England in 1914. Ramanujan's reluctance to go England without his mother's permission was a typical Hindu reaction. It is a common practice in India even today to take the blessings of elders before embarking on a voyage.

While in England, the rise of Ramanujan's reputation was meteoric. In frequent discussions with Hardy he was showing several new results each time. For example, under Ramanujan's magic hand, the Theory of Partitions underwent a glorious transformation. He discovered several astonishing new theorems on partitions involving congruences and continued fractions. In collaboration with Hardy he showed how to obtain an accurate formula for the number of partitions of an integer. This is the famous Circle Method so widely used in Number Theory today. In another paper with Hardy, he began the investigation of round numbers which led to the creation of Probabilistic Number Theory several years later by mathematicians like Paul Erdös. His research record was so impressive that he was elected Fellow of The Royal Society (F.R.S.) in 1918.

Ramanujan, who was a gregarious and orthodox Brahmin, found himself in an awkward position amidst educated Englishmen who were so aloof. Socially, Hardy was the opposite of Ramanujan. This book is a dual biography—of Ramanujan and Hardy. And the author succeeds wonderfully in showing the gulf that separated the two. What bridged this gap was mathematics, but here too there was considerable difference in the way they thought. Ramanujan was a genius who conjectured and made giant leaps of imagination; as a seasoned mathematician, Hardy put emphasis on rigor and proceeded by logical step-by-step reasoning. Kanigel's description of Hardy's British upbringing is superb, but he spends too much time analyzing Hardy's lack of interest in women.

Healthwise, England was a disaster for Ramanujan. He never got adjusted to the cold weather. He was in and out of sanatoria being treated mainly for tuberculosis. Having been used to the curries and spices of India, he found English food to be tasteless. In 1919 his health became so bad that he returned to India. He died in Madras in April 1920.

Hardy felt that the real tragedy was not Ramanujan's early death, but that in India, he wasted much time in rediscovering past work. He argued that the best creative work is done when one is very young and so, at the time of his death, Ramanujan was perhaps past his prime. But here Hardy may have been wrong. Ramanujan's now famous work on the Mock Theta Functions was done during his last few months in India. He wrote one last letter to Hardy summarizing his discoveries. They are now considered to be among his deepest contributions. So Ramanujan was definitely on the rise, and he could have reached even greater heights had he lived longer.

Hardy compared Ramanujan to Euler and Jacobi as a genius. Yet he was of the opinion that Ramanujan's work was strange and lacked the simplicity of the very

greatest works. With recent advances in the Theory of Modular Forms and the research of Andrews on Ramanujan's Lost Notebook, we now realize that Ramanujan's work is more fundamental than what Hardy had imagined. Ramanujan's equations are now being used to compute *pi* (the ratio of the circumference of a circle to its diameter) to a billion digits! Atle Selberg of the Institute for Advanced Study, Princeton, has said that it will take many more decades, possibly more than a century, to fully understand Ramanujan's contributions.

The fascinating story of discovery of the Lost Notebook is described in this book. Shortly after Ramanujan's death, his widow Janaki collected all the loose sheets on which Ramanujan had scribbled mathematics and sent them to Hardy. They contained over six hundred formulae including many on mock theta functions. Hardy handed this manuscript to G.N. Watson who wrote two papers on this topic. After Watson's death this manuscript was placed along with Watson's papers at the Wren Library in Cambridge University, and the mathematical world remained unaware of its significance. In 1976 Andrews stumbled across the manuscript while doing some reference work. He instantly recognized it as a priceless treasure and has been analyzing its contents ever since.

When Ramanujan's centenary was celebrated in India in December 1987, mathematicians from all over the world came to pay homage to this legendary genius. There were several conferences held in India of which two were in Madras. The first of these was at Anna University, for which Andrews had come in connection with a session on Number Theory that I organized. Mrs. Ramanujan, who was 87 years old, was present on the opening day. Kanigel says: "Andrews, his voice choked with emotion, presented Janaki with a shawl. It was she who deserved the credit for the Lost Notebook, he said, since it was she who kept it together while Ramanujan lay dying." At the second conference, India's Prime Minister, Rajiv Gandhi, released two copies of The Lost Notebook and presented one to Mrs. Ramanujan and the other to George Andrews.

Bruce Berndt is editing the notebooks of Ramanujan. He has published three volumes, and two more are forthcoming. Andrews and Berndt have plans to edit the Lost Notebook. Owing to the efforts of Andrews, Askey, and Berndt, it is now possible to include Ramanujan's work as part of the regular graduate mathematics curriculum. And, by reading this fascinating biography, students will be drawn to a study of Ramanujan's spectacular results.

Chapter 21
A Review of "Ramanujan: Letters and Commentary"

Srinivasa Ramanujan is a truly exceptional figure in mathematical history. Born in a poor and orthodox Hindu family in rural India in 1887, Ramanujan did not even have a college degree. Yet, he made startling mathematical discoveries that challenged the finest minds of his and succeeding generations. Ramanujan communicated some of his remarkable findings in letters to G.H. Hardy of Cambridge University who was so impressed that he helped to make arrangements for Ramanujan to go to England. During the brief period of five years (1914–1919) that he spent in England, Ramanujan wrote several fundamental papers, some in collaboration with Hardy, which revolutionized certain areas of mathematics. His work was considered so original and profound that he was made Fellow of The Royal Society (FRS) and Fellow of Trinity College. Unfortunately, the rigors of life in England during World War I combined with his own peculiar habits contributed to a decline in his health. This forced him to return to India in 1919 where he died a year later. Despite his early death, the papers that he wrote in England and in India and the mass of unpublished material that he left behind in the form of two notebooks and loose sheets of paper have had a deep and lasting impact and continue to inspire researchers today.

Ramanujan's story is sad, yet awe inspiring in that someone who was so poor financially, who did not have a college education, and who grew up in such an old fashioned society, defied all odds and reached the pinnacle in mathematics, the most abstract and rigorous of all disciplines. Much has been written about Ramanujan and justifiably so, for he and his work are worthwhile studying from many points of view. For the layman, there is the wonderful book by Kanigel [3] describing Ramanujan's incredible life story. For those with a mathematical background, there is an account of Ramanujan's work by Hardy [2] in the form of twelve lectures, considered a classic in exposition. Then there are the Collected Papers [4], where at the beginning there are two charming biographies of Ramanujan written in contrasting

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styles, one by Hardy and another by P.V. Seshu Aiyar and R. Ramachandra Rao. For those who wish to delve directly into Ramanujan’s identities and drink delight of his discoveries, there are the photostat versions of his two Notebooks [5] and The Lost Notebook [6] as well. And to top this all, there is a series of five volumes by Bruce Berndt [1] thoroughly discussing the hundreds of “entries” made by Ramanujan in his famous notebooks, not to mention many research papers written in this century dealing with Ramanujan’s work, in particular by George Andrews. So what could be added of great significance to this already impressive collection? As Shakespeare remarked, would it be like gilding refined gold, painting the lily, or adding another hue to the rainbow? Not at all, as this book under review demonstrates.

By assembling letters to, from and about Ramanujan, and by giving an authoritative commentary on these letters, Berndt and Rankin have produced a book that should appeal to everyone with an interest in mathematics. Many of the letters contain mathematics. For these letters, Berndt and Rankin explain the results and their significance with appropriate references. There are also letters from persons who knew Ramanujan and had an impact on his life. For such letters there are explanations of the connections these persons had with Ramanujan or his work. By reading the letters along with the comments, we get a better understanding of Ramanujan’s personality, his life, and his remarkable work.

This book opens with a series of letters which describe how Ramanujan got a job at the Madras Port Trust and the efforts made by various well wishers to help him pursue his mathematical investigations. Although Ramanujan held a scholarship in his school, owing to his excessive preoccupation with mathematics, he neglected other subjects and consequently failed his college examinations in all subjects except mathematics. Naturally, without a college degree, he could not secure a job to support himself so that he could continue to do mathematics unhindered by financial difficulties. His family being so poor was in no position to support him financially. Therefore Ramanujan approached several persons for help and showed them some of the remarkable identities recorded in his notebooks. These persons included Indians in well placed positions as well as British administrators and professors in educational institutions in the city of Madras. While every one of them was convinced of Ramanujan’s unusual abilities and creativity, no one was able to judge the value of Ramanujan’s work or understand it in the proper mathematical framework. One gentleman, R. Ramachandra Rao, recognized that Ramanujan was doing very original work and gave him some financial support. It was clear that Ramanujan ought to come into contact with first rate research mathematicians. With this favorable intention in mind, C.L.T. Griffith, Professor of Civil Engineering in Madras wrote to M.J.M. Hill of the University College, London about some of Ramanujan’s work. Hill realised, as others had, that Ramanujan was very gifted, but when he saw Ramanujan’s outrageous claims that

$$1+2+3+\cdots=\frac{-1}{12}, \tag{21.1}$$

and

$$1^2+2^2+3^2+\cdots=0, \tag{21.2}$$

he felt that Ramanujan had bungled. And in a reply to Griffith, Hill expressed the opinion that these are the kind of blunders one would make without a formal mathematical training. After all, Abel has warned us that "divergent series are in general deadly, and anyone who dares to base a proof on them is doomed to failure". We know now that Ramanujan had not bungled here. On the contrary, these are significant assertions if viewed as follows. Consider the Riemann zeta function which is defined by the series

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}, \tag{21.3}$$

for $Re(s) > 1$. It is known, and this is a very significant fact, that $\zeta(s)$ admits an analytic continuation as a meromorphic function throughout the complex plane having $s = 1$ as its only singularity. Using this analytic continuation one can show that

$$\zeta(-1) = \frac{-1}{12} \quad \text{and} \quad \zeta(-2) = 0. \tag{21.4}$$

Observe that by setting $s = -1$ and $s = -2$ formally in (21.3) and using the values given in (21.4), the two claims made by Ramanujan in (21.1) and (21.2) follow.

But Ramanujan did not explain his claims in this fashion. In fact he gave no explanation at all and that is what stunned Hill. We know now that Ramanujan had his own theory of infinite series. In particular, to each series, whether it is convergent or not, he associated a "constant". The constant he associated with the series in (21.1) was $-1/12$ and the constant he assigned the series in (21.2) was 0. Although no one at that time had any understanding of Ramanujan's methods, it was clear that he was unusually talented. So he deserved to have a scholarship or other financial support to enable him to continue his investigations. Since Ramanujan did not have a college degree, arranging a university scholarship was next to impossible, and so a job was initially secured for him in March 1912 as a clerk in the Accounts Department at the Port Trust in Madras with the help of its Manager, S. Narayana Aiyar. The Chairman of the Port Trust, Sir Francis Spring, and Narayana Aiyar took a keen interest in Ramanujan's work and later helped him to secure a scholarship at the Madras University and also in preparations for his trip to England. In fact, Narayana Aiyar even worked with Ramanujan on some mathematical problems.

On February 13, 1913, Ramanujan wrote a letter to G.H. Hardy of Cambridge University. This letter is one of the most important and exciting mathematical letters ever written! After begging to introduce himself as a poor clerk in the Port Trust of Madras, Ramanujan gives a collection of amazing results he had obtained. These included representations in the form of series and integrals for the number of primes up to a given magnitude and related arithmetical functions, identities for beta and gamma functions, infinite series identities having arithmetical significance as well as consequences in the theory of elliptic and theta functions, and some unbelievable continued fraction evaluations in terms of certain algebraic numbers. There were some results in the letter that were well known and some that were wrong. But then, there were also many that were startlingly new and very deep. Hardy could prove a

few of these, but there were others that defeated him completely. And Ramanujan had not given any hints as to how he went about proving them!

Hardy showed the letter to his colleague J.E. Littlewood and the two came to the conclusion that Ramanujan was a mathematician of the highest caliber and a genius on par with Euler and Jacobi in manipulative ability! Hardy was very excited about this and soon the news spread through Cambridge that at least another Jacobi in the making had been found. There is a very nice letter from Bertrand Russell to his sweetheart Lady Ottoline Morrell that is reproduced in this book wherein Russell says he "found Hardy and Littlewood in a state of wild excitement, because they believe they have discovered a second Newton". In replying to Ramanujan, Hardy pointed out the results that were wrong, those that were well known, and those which were new and most impressive. But he insisted that Ramanujan should supply proofs of his results. In his second letter to Hardy which contained many more results, Ramanujan says that the reason he did not describe his methods was because they were so unusual, that persons with formal training may not appreciate these unconventional approaches as had been his experience with Hill. But Ramanujan was confident in the validity of his methods because in his first letter to Hardy he actually said—"the local mathematicians are unable to understand me in my higher flights". Both letters of Ramanujan to Hardy are reproduced in full here with comments on their mathematical contents.

Hardy was convinced that Ramanujan should not waste any more time in India but should come to England where his untutored genius would develop its full potential and be given a proper sense of direction. Thus Hardy urged Ramanujan to come to England. Ramanujan initially was reluctant to go abroad owing to his religious background and pressure from his family, but eventually agreed to go for the sake of mathematics.

During the five years that Ramanujan spent in England he wrote several papers which revealed the range and power of his mathematics. In collaboration with Hardy he wrote two great papers. In the first one that appeared in 1917, he studied the most commonly occurring values of the number of prime factors of an integer. What is surprising here is that, although prime numbers have been studied since Greek antiquity and various arithmetical functions have been investigated in the subsequent centuries, it was the first systematic discussion of the number of prime divisors of an integer. This paper led to the creation of Probabilistic Number Theory. In another joint paper with Hardy published in 1918, he gave an asymptotic formula for $p(n)$, the number of partitions of an integer n, using the *circle method*. This powerful analytic technique is now the standard tool in Additive Number Theory and its genesis may be traced to a formula that Ramanujan gave for the coefficients $c(n)$ of a certain series in his first letter to Hardy. What Hardy and Ramanujan produced was a series representation for $p(n)$ with terms involving the exponential function. Summing the series up to a certain number of terms dependent on n yielded a value whose nearest integer was $p(n)$. Later, Hans Rademacher made the important observation that replacing the exponential function by suitable hyperbolic functions converts the Hardy-Ramanujan representation into an infinite series which converges to the value $p(n)$. Subsequently D.H. Lehmer showed that the Hardy-Ramanujan representation

in terms of the exponential function is actually divergent as an infinite series. The correspondence between Hardy and Lehmer concerning the partition function is included in this book. It is worth noting that the formula that Ramanujan communicated Hardy concerning $c(n)$ made use of hyperbolic functions as in Rademacher's formula.

In England, Ramanujan also wrote many fundamental papers by himself. In one paper he established unexpected congruence properties for the partition function. After all, partitions represent an additive process and so it is surprising that certain divisibility (congruence) properties are valid here. In yet another famous paper on Modular Equations and Approximations to π, he gave several astonishing series representations which are being used in this modern era of computers to calculate the digits of π. This paper has also provided new insights into the properties of elliptic and modular functions.

Although his mathematical productivity was great, Ramanujan never got adjusted to the British way of life. The English winters were too harsh for him and he did not know how to adequately protect himself from the cold. He was a strict vegetarian and so did not want to eat the food served in the dining hall at Trinity College, preferring to cook his own food. But he was not very good at this. In particular, he did not eat a balanced diet. Things were made worse by the wartime rationing in England. All this had a catastrophic effect on his health. He was in and out of several sanatoria and hospitals because he was suspected to have tuberculosis. Even today we are not sure what exactly was Ramanujan's ailment. There is good reason to believe that he was suffering from hepatic amoebiasis, a parasitic infection of the liver or intestines that is found in tropical countries. There are many letters in this book that deal with Ramanujan's health problems.

Ramanujan's work was so original and important that Hardy felt that the Indian genius deserved to made Fellow of The Royal Society and Fellow of Trinity. With Ramanujan's health declining rapidly, Hardy wanted these honors to be conferred without delay. Hardy also felt that this would boost Ramanujan's spirit and have a positive effect on his health. So Hardy moved heaven and earth and finally succeeded in having these honors conferred on Ramanujan. This book contains a wonderful collection of letters between Ramanujan and his relatives and friends in India, wherein Ramanujan describes his mathematical successes in spite of the difficulties of living in England. In these letters he describes the particular Indian vegetarian dishes he is able to prepare. By reading the commentary of Berndt and Rankin, one gets a crash course on the curries and spices of South India! The correspondence between Ramanujan and Hardy while Ramanujan was in nursing homes is also included. It is amazing to see the quality of mathematics that Ramanujan communicated to Hardy from hospital beds in England! Finally, there are letters between Hardy and his British contemporaries giving us an idea of the efforts that went in to get Ramanujan elected Fellow of The Royal Society and Fellow of Trinity.

But Ramanujan's health did not improve substantially as Hardy and others had hoped. As soon as he was well enough to travel, Ramanujan returned to India in 1919. In India, his health worsened, and he died on April 26, 1920. But his mathematical powers had not diminished even in his final moments. He wrote one last

letter to Hardy in which he described his latest discovery, the mock-theta functions, and expressed the opinion that these enter mathematics more naturally and beautifully than the false-theta functions of L.J. Rogers. These are now considered to be among Ramanujan’s deepest contributions.

After Ramanujan’s death, Hardy received the loose sheets of paper on which Ramanujan had scribbled mathematical formulae in his dying moments. We should be grateful to Janaki Ammal, Ramanujan’s widow, for not throwing away his final jottings. Later Hardy handed these sheets to G.N. Watson at Birmingham. Contained in these sheets were several deep formulae for the mock-theta functions. Watson wrote two papers on the mock-theta functions but there was much more in these loose sheets that needed to be analyzed. After Watson’s death, all the mathematical papers in his house were collected and deposited at the Wren Library in Cambridge University. Included in this collection were the last writings of Ramanujan. Surprisingly, the mathematical world remained unaware of the existence of these notes of Ramanujan until George Andrews accidentally came across them in 1976 while looking through the Watson collection at the Wren Library. The fascinating story of the discovery of Ramanujan’s Lost Notebook by Andrews has been told many times. During the Ramanujan Centenary Celebrations in Madras, India, the Lost Notebook and other unpublished papers of Ramanujan as well as some correspondence between Hardy, Watson and others after Ramanujan’s death was brought out in a printed form [6]. These letters along with commentaries are included in this book.

At the beginning of this century, Ramanujan was perceived by Hardy as a genius of the first magnitude who, without a sense of direction, had unfortunately wasted some of his best years rediscovering past work in India. Of course, Hardy did acknowledge that there was a significant number of new results in Ramanujan’s work and that there were many which were so singularly original that he ranked among the great mathematicians in history. Today we are in a much better position to comprehend the grandeur and significance of Ramanujan’s contributions. Hardy had said that Ramanujan should have perhaps been born a century earlier during the great days of formulae. But Richard Askey points out that currently in physics there are incredible formulae in several variables that are being analyzed and a genius like Ramanujan would be of invaluable help. As Askey puts it, “the great age of formulae may be over, but the age of great formulae is not!”

Speaking about Ramanujan, Hardy described him as “a poor, uneducated Hindu pitted against the accumulated wisdom of Europe”. At this point I would like to compare Ramanujan with another equally astonishing figure, also from India, not in mathematics but in politics, namely, Mahatma Gandhi. Described by Sir Winston Churchill as “the half-naked fakir”, Gandhi, wearing only a home spun loin cloth, stemmed the tide of British imperialism by his simplicity, honesty and adherence to non-violence. In a political world of charisma and diplomacy, who would have expected Gandhi to be a success? But Gandhian philosophy has had a lasting impact just as Ramanujan’s equations scribbled on a piece of slate or on loose sheets of paper influence the work of some of the most sophisticated mathematicians today. Those turning to Ramanujan for inspiration would find in this book a fine introduction to the type of mathematics that Ramanujan did and a good sample of his great

discoveries with references to over three hundred research papers, books and articles. Those eager to learn about Ramanujan's life would get as a bonus, a glimpse into the culture and traditions of the Hindu way of life in British Colonial India. And finally, what better way to understand the man behind the mathematician Ramanujan than to read letters written by him and about him? Berndt, with the experience he has gained editing Ramanujan's notebooks, and Rankin, one of the veterans in this field who knew Hardy, Littlewood, Watson and other British contemporaries of Ramanujan, have combined perfectly to produce this book.

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Chapter 22
A Review of "Ramanujan: Essays and Surveys"

How much can be written about the Indian mathematical genius Srinivasa Ramanujan? Quite a bit for a variety of reasons. First, Ramanujan's contributions are astonishing, and with time we are getting a better understanding of their significance and depth. Next, the fact that Ramanujan made such spectacular discoveries without any formal education makes one wonder how his brilliant mind must have worked and what motivated him to make these discoveries. Finally, Ramanujan's life story is a mixture of success and failure, (mathematical) romance and sadness, and a blend of the mystery of the East with the sophistication of the West. All these make him one of the most intriguing and fascinating characters to study. Thus there is a multitude of articles describing various aspects of his life, and research papers that build on his path-breaking discoveries. The book under review assembles a fine collection of essays on Ramanujan by several distinguished authors on a variety of topics ranging over his remarkable life and contributions, his constant struggle with various illnesses in India and in England, none of which dulled his ferocious productivity, his manuscripts and notebooks, as well as articles on individuals who played an important role in Ramanujan's life and rendered timely help.

They say a picture is worth a thousand words! The most well-known photograph of Ramanujan is his passport photo taken when he sailed to England in 1914 to work with G.H. Hardy in Cambridge University. After Ramanujan's death in 1920, Hardy wanted a photograph of Ramanujan in connection with the publication of his Collected Papers. The young astrophysicist S. Chandrasekhar was going to India, and Hardy requested him to secure a photograph of Ramanujan. Chandrasekhar met Ramanujan's widow Janaki and was delighted to find that she had preserved the passport photograph of her late husband. Upon seeing this picture Hardy said "He looks rather ill, but he looks all over the genius he was." It was this picture that formed the basis for the American sculptor Paul Granlund to make busts of Ramanujan in

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1987 during the Ramanujan Centennial Year. Incidentally, Chandrasekhar became one of the most influential astrophysicists of the twentieth century and won the Nobel Prize in 1983. This book opens with a set of four photographs of Ramanujan and an analysis of each photograph, including Chandrasekar's own description of how he played a role in obtaining the passport photograph from Mrs. Ramanujan.

One of the books that inspired Ramanujan was *A Synopsis of Elementary Results in Pure Mathematics* by G.S. Carr. As Hardy said, this was not a great book in any sense, but really a set of mathematical formulae arranged in a certain logical sequence and with little or no proofs. Carr was a tutor in England, and he used this book to train his students. This was one of the first books Ramanujan set eyes on and was immediately inspired to prove the formulas there and discover similar formulas. Indeed Ramanujan recorded his own discoveries in two Notebooks in the style of Carr by writing down identities one after the other with no hints of proofs.

Hardy expressed the opinion that Ramanujan did not have a grasp of complex variable theory and that this was the cause for some of the slips that Ramanujan made in the theory of prime numbers. What is most baffling is that Ramanujan had a complete mastery over elliptic and theta functions, a subject in which significant contributions cannot really be made without a firm grasp of complex variable theory. Ramanujan's theory of elliptic functions was work that he did in India prior to his departure to England. Hardy was of the belief that Ramanujan did not invent elliptic functions by himself, that he must have had access in India to Greenhill's book or other books on these topics from which he must have learnt some of the basic ideas. Of Ramanujan's remarkable ability to evaluate elliptic and other definite integrals, Hardy has said that during the course of his lectures if at any time he needed the value of a certain integral, he would simply turn towards Ramanujan in the audience who would provide the answer instantly! We probably can never answer satisfactorily how Ramanujan arrived at his formulas because he thought so differently from others with a more conventional training. But we can at least attempt to understand how Ramanujan might have been led to his remarkable findings from a list of books he read. Included in this book is an article by Berndt and Rankin on the books Ramanujan read in India with an analysis of these books and the influence they might have had on the Indian genius. Hardy's book *Orders of Infinity* also attracted Ramanujan who was intrigued by the problem of determining the number of primes below a given magnitude, and indeed it was this that prompted Ramanujan to write to Hardy. The rest is history. The collaboration between Hardy and Ramanujan, the influence they had on each other, the impact their work has had over mathematicians of their generation and those succeeding them is immense.

Ramanujan made spectacular contributions to several areas of mathematics at the interface between analysis and number theory such as elliptic and modular functions, continued fractions, definite integrals, hypergeometric series, etc., and often his results revealed unexpected connections between fields considered quite distinct. His results are recorded in two notebooks that he maintained in India, and in various papers he published in mathematical journals while in England (1914–1919) and in India prior to his departure to England. In this book there are two thorough articles by Berndt and Rankin on the two notebooks of Ramanujan and his various

manuscripts. After return from England in 1919, Ramanujan was dying, but even in failing health he did not stop doing mathematics. His wife Janaki supplied him loose sheets of paper on which he wrote down several beautiful identities involving mock theta functions. He wrote a letter to Hardy in 1920, just weeks before his death, saying that he had found a new important class of functions called mock theta functions and gave some examples of these. Ramanujan's work on mock theta functions are considered now to be among his deepest contributions. After Ramanujan's death, his widow Janaki had the good sense to collect these loose sheets containing mathematical formulae, and realising that they were of mathematical significance, she sent them to Hardy. Hardy handed over these last writings of Ramanujan to G.N. Watson in Birmingham and asked him to analyse them. In his Retiring Presidential Address to the London Mathematical Society, Watson gave a lecture entitled *The Final Problem: An Account of the Mock Theta Functions*. Watson said: "The study of the five foolscap pages which accompanied the (Ramanujan) letter is the subject which I have chosen for my address. I doubt whether a more suitable title (for my talk) could be found than the title used by John H. Watson, M.D., for what he imagined to be the final memoir on Sherlock Holmes." Watson's lecture is reproduced in full in this book.

After Watson's death, the Ramanujan manuscript mysteriously disappeared. So it acquired the name *The Lost Notebook of Ramanujan*. In 1976, George Andrews, one of the greatest experts in the theory of partitions and q-series and on the work of Ramanujan, stumbled across this manuscript while going through the Watson estate at the Wren Library of Trinity College in Cambridge University. Andrews immediately realised the invaluable treasure he had found. He wrote an article in the American Mathematical Monthly that year giving a glimpse of the beautiful formulae in the Lost Notebook. This beautiful article is also reproduced in this book.

In order to gain a complete understanding of Ramanujan, one must also learn about those individuals who played a crucial role in his life and offered timely help. The authors are to be complemented for including biographies of Mrs. Ramanujan and Narayana Iyer.

Born in 1900, Janaki was wedded to Ramanujan when she was a child. When Ramanujan left for England in 1914, she was barely a teenager. When Ramanujan returned to India in 1919, he was a very sick man. She attended on him with affection and devotion. Although she was uneducated, she was a very intelligent woman, and realised that her husband was an unusual genius doing work of great importance. Owing to this good sense, she preserved whatever Ramanujan wrote in his last days and mailed them to Hardy. I remember well in 1987 during the Ramanujan Centennial in Madras, George Andrews paying a wonderful tribute to Mrs. Ramanujan, who was present for the occasion. In a voice choked with emotion, Andrews thanked Mrs. Ramanujan for preserving the pages of the Lost Notebook. The story of Janaki Ramanujan is sad. Yet she played a crucial and positive role in Ramanujan's life.

S. Narayana Iyer was the Chief Accountant of the Madras Port Trust Office where Ramanujan worked for over a year. He was perhaps Ramanujan's closest mathematical friend in India and supported Ramanujan's mathematical research.

Ramanujan published an important paper in the Oxford Quarterly Journal of Mathematics (1914) entitled "Modular equations and approximations to π". This

paper contains a myriad of formulae including transformation formulas for elliptic and theta functions called modular relations. In fact, Ramanujan discovered more modular relations than Abel, Jacobi, and other luminaries combined! Ramanujan's approach to elliptic and theta functions is so original, and his notation so different from that of his illustrious predecessors, that contrary to Hardy's opinion, one is inclined to believe that Ramanujan discovered these results without prior knowledge of the subject.

In addition to modular identities, this paper contains several series representations for the reciprocal of π and for numbers which are of the form π divided by the square root of an integer. Since these series converge very rapidly, it was realized that they could be used to calculate the digits π and other numbers. William Gosper used one of Ramanujan's series for the reciprocal of π to evaluate 17 million terms in the continued fraction expansion of π. More recently, the brothers David and Gregory Chudnovsky utilized certain extensions of some of Ramanujan's formulae to compute π to about two billion decimal places. The most outstanding thing about this calculation was that the Chudnovsky brothers did this by assembling a computer (by mail order) in their own apartment in New York, a computer built specifically for this purpose! In contrast, it is amazing that Ramanujan, who in rural India wrote many of these formulae on a slate and erased them with his elbow, should remain alive in the modern world of the computer! Incidentally, in this book there is a picture of Bruce Berndt holding this slate of Ramanujan.

Many may wonder what is achieved by calculating millions of digits of π. Is it simply for the challenge? Of the calculation of π to 15 decimal places, Newton admitted "I am ashamed to tell you to how many figures I carried out these computations, having no other business at this time". Sir Edmund Hillary's response when asked why he chose to climb Mt. Everest was "because it is there".

In the case of the calculation of the digits of π there is more at stake than just the challenge. Every attempt to understand π has led to new techniques that have applications elsewhere. Often various iterative algorithms are tested by using them to compute the digits of familiar numbers like π. In an article entitled *Ramanujan and Pi*, Jonathan and Peter Borwein have given a detailed account of the historical developments relating to our understanding of π and about the usefulness of many of Ramanujan's formulas in the calculation of π. This magnificent article is also reproduced in this book.

There are other beautiful and important articles in this book such as *A walk through Ramanujan's garden* by Freeman Dyson and *Ramanujan and hypergeometric series* by Richard Askey. Dyson's article is basically the text of a speech he gave at the Ramanujan Centennial Conference held at the University of Illinois, Urbana, in the summer of 1987.

I will conclude by briefly discussing some of the contents of an article by Selberg entitled *Reflections around the Ramanujan Centenary* that Berndt and Rankin chose to include in this book. This was the talk given by Atle Selberg, one of the greatest mathematicians of the twentieth century, on 22 December 1987 (Ramanujan's one hundredth birthday), in Madras, India. I was fortunate to be in the audience and listen to this thought provoking lecture.

Selberg begins by recalling how as a boy he read an article on the Indian genius Ramanujan and how various astonishing identities of Ramanujan inspired him to study similar questions. In particular, Selberg was intrigued by the famous Hardy–Ramanujan asymptotic series expansion for $p(n)$, the number of partitions of n. As was always the case when Selberg attempted to understand a problem, he set about deriving the Hardy–Ramanujan formula himself. In doing so, he actually ended up with a superior convergent series expansion for $p(n)$. Selberg did not publish this when he found out that Hans Rademacher had already obtained this convergent series.

In one of the letters to Hardy in 1913, Ramanujan gave a formula for the coefficients of a certain series expansion of an infinite product which suggested that there ought to be a similar exact formula for $p(n)$. Hardy felt that this claim of Ramanujan concerning an exact formula for $p(n)$ in terms of continuous functions was too good to be true but was convinced that it would be possible to construct an asymptotic series expansion. By an ingenious and intricate calculation involving the singularities of the generating function of $p(n)$, Hardy and Ramanujan obtained an asymptotic formula which when calculated up to a certain number of terms yielded a value that differed from $p(n)$ by no more than the fourth root of n. Since $p(n)$ is an integer, it is clear that its value is the nearest integer to what is given by the series. The series Hardy and Ramanujan obtained was genuinely an asymptotic series in the sense that when summed to infinity, it diverges. Subsequently, Hans Rademacher noticed that by making a very mild but important change, namely, by replacing the exponential functions by hyperbolic functions, the Hardy–Ramanujan asymptotic series could in fact be converted to an infinite series that converges to $p(n)$. Actually in the 1913 letter to Hardy, Ramanujan used hyperbolic functions to claim an exact formula for a related problem. So Ramanujan was indeed correct in surmising that a similar exact formula would exist for $p(n)$. According to Selberg, even though Ramanujan claimed the existence of an exact formula for $p(n)$, out of respect for his mentor Hardy, he settled for less, namely the asymptotic formula.

Selberg also expresses the opinion in this article that the great German mathematician Hecke would have been a better mentor for Ramanujan than Hardy. Selberg's observation is based on the fact that many of Ramanujan's important discoveries lay in the area of modular forms, a field in which Hecke was an expert, whereas Hardy was not. But there are other factors that are crucial in mentoring, such as effective communication in a common language, willingness to spend time with a pupil, and above all mutual respect. Hardy was indeed an outstanding mentor. Whether or not one agrees with Selberg on this opinion, this article, which provides the reader several insights, is a magnificent tribute to Ramanujan.

This book is a worthy sequel to *Ramanujan—Letters and Commentary* [2] (see my review [1]). It is amazing that the authors have once again produced a book that is identical in length, 347 pp.! The present book also nicely complements the classic *Ramanujan: Twelve Lectures on Subjects Suggested by His Life and Work* [3] and Ramanujan's *Collected Papers* [4]. Sadly, the second author Robert Rankin died on 27 January 2001, soon after the manuscript form of this book was finished.

In his charming article *A walk through Ramanujan's garden*, Freeman Dyson says that any time he felt angry or depressed, he would pull down Ramanujan's

Collected Papers from the bookshelf and take a quiet stroll in Ramanujan's garden. He recommends this therapy to all who suffer from headaches and jangled nerves. Similarly, this book with its delightful collection of essays and surveys relating to Ramanujan's life and contributions is accessible to, and an inspiration for, laymen, students, and professional mathematicians.

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Chapter 23
A Review of "Partition: A Play on Ramanujan"

While I was in San Francisco during May 2–5 to give a talk at an American Mathematical Society Meeting, I had the opportunity to see a play at the Aurora Theatre in Berkeley entitled PARTITION. This is about the Indian mathematical genius Srinivasa Ramanujan.

The playwright is Ira Hauptman, and the cast consisted of six characters: Ramanujan, Professor G.H. Hardy of Cambridge University who was Ramanujan's mentor, The Goddess of Namakkal who appears in Ramanujan's dreams and gives him wonderful mathematical formulas, Billington who is a (fictional?) colleague of Hardy's at Cambridge University, a police officer from Scotland Yard, and Pierre Fermat, a famous French Mathematician of the 17th century.

The play begins with a scene at Scotland Yard where Ramanujan is being questioned by a police officer for attempting to commit suicide by throwing himself on the subway tracks. Ramanujan says that he wanted to die because he inadvertently ate meat and therefore had sinned. Hardy tells the police officer that Ramanujan is a Fellow of the Royal Society (FRS)—which he was not at that time, but only later—and gets him released. So that sets the stage for the introduction of this unusual personality from India who is a genius of first magnitude.

The next scene of the play is a discussion between Hardy and Billington about the letter from Ramanujan which contained several incredible mathematical formulae, and the decision by Hardy to invite Ramanujan to Cambridge. The play then proceeds with Ramanujan's arrival in England to work with Hardy. There is a scene which focusses on Ramanujan's remarkable insight which led to the Hardy–Ramanujan asymptotic formula for partitions, hence the title of the play.

Fermat is brought into the play in a somewhat surprising manner. As is well known, Fermat's Last Theorem is his assertion that the equation $x^n + y^n = z^n$ has no solutions in positive integers x, y, z if the exponent n is at least 3. This statement by Fermat gained fame because when he recorded it in his notebook, he made the claim that he had a "truly marvellous" proof and that the margin of his notebook was

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too small to contain it! Fermat’s Last Theorem resisted attempts for a solution for three centuries, and it was only in the 1990s was this great problem solved. In the play Hardy suggests that Ramanujan should work on Fermat’s Last Theorem. This is really the playwright’s fancy. There is even a conversation in the play between Ramanujan and the Goddess of Namakkal concerning Fermat’s Last Theorem. As far as we know, Ramanujan never worked on Fermat’s Last Theorem nor did Hardy suggest that he should. Fermat’s character is quite impressive, and so his introduction in the play for the sake of effect is acceptable artist’s fantasy.

No account of Ramanujan is complete without the Taxi Cab episode. There is a charming scene of Hardy meeting Ramanujan in a hospital, and when Hardy mentions that he arrived by the taxi numbered 1729, Ramanujan immediately points out that $1729 = 10^3 + 9^3 = 12^3 + 1^3$, the smallest positive integer that can be expressed as a sum of two cubes in two different ways. Of course the Ramanujan taxi-cab equation $x^3 + y^3 = z^3 + w^3$ yields Fermat’s equation for cubes by setting $w = 0$, but it is to be noted that the taxi-cab equation has positive integer solutions, whereas Fermat’s does not. In 1995 I wrote an article for The Hindu entitled “Fermat and Ramanujan—a comparison” following the announcement of Andrew Wiles and Richard Taylor that the proof of Fermat’s Last Theorem was completed. In that article I pointed out that although Ramanujan never worked on Fermat’s Last Theorem, there are several similarities between Fermat and Ramanujan. For instance both recorded their findings in notebooks, both communicated their discoveries in letters, and both provided only sketches of proofs of their claims in many instances.

The play concludes with the letter Hardy receives from India conveying the sad news of Ramanujan’s death, and Hardy’s address to the London Mathematical Society giving an account of the spectacular contributions of Ramanujan.

In the past few decades we have witnessed how Ramanujan’s contributions have made such a profound impact on various branches of mathematics. The book *The man who knew infinity* by Robert Kanigel reached out to the general public the world over by describing the fascinating life story of Ramanujan. And now, in the form of a play, the public is made aware, once again, of this wonderful story. This is a very impressive play, and I had the pleasure of seeing it with Professor George Andrews, the world’s greatest authority on Ramanujan’s work and on partitions.

I heard about the play only upon arrival in San Francisco. When Professor Andrews called the Aurora theatre to check for ticket availability, he was told there were just two seats left, and so we took them. We could not believe our luck because it was Saturday evening and we called just a few hours before the play. But I had a feeling that the Goddess of Namakkal had preserved these seats for us!

Chapter 24
The Ramanujan Journal: Its Conception, Need and Place

In January 1997, *The Ramanujan Journal*, a new international journal, will be launched by Kluwer Academic Publishers of The Netherlands. This is perhaps the most significant event involving the international mathematical community in the continuing Ramanujan saga since the Ramanujan centennial in 1987. This journal is devoted to all areas of mathematics influenced by Srinivasa Ramanujan. The Editorial Board consists of 27 leading mathematicians from around the world including Professors George Andrews, Richard Askey and Bruce Berndt, the three greatest experts on Ramanujan's work. The launching of the journal is a further testimony for the high regard that the international mathematical community has for Ramanujan and for the enormous impact that his work has made. In this article I shall describe the conception and evolution of the Ramanujan Journal. In doing so, I will discuss its aims and scope and compare this with those of certain other leading mathematics journals.

Conception and Evolution Professor G.H. Hardy of Cambridge University compared Ramanujan to Euler and Jacobi for sheer manipulative ability. Over the years our understanding of the grandeur of Ramanujan's contributions has increased considerably, and we realise now that his work has a wider impact than was originally imagined. Today, the name Ramanujan brings to mind a whole range of topics including:

(i) Hypergeometric and basic hypergeometric series (q-series)
(ii) Partitions, compositions and combinatory analysis
(iii) Circle method and asymptotic formulae
(iv) Elliptic and theta functions
(v) Mock theta functions
(vi) Modular forms and automorphic functions

This article is based on a talk given at the M.S. Swaminathan Foundation on 30 December 1996 and appeared in *The Hindu*, India's National Newspaper, on 17 January 1997. The Ramanujan Journal is now published by Springer.

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(vii) Special functions and definite integrals
(viii) Continued fractions
(ix) Diophantine analysis
(x) Number theory
(xi) Fourier analysis with applications to number theory
(xii) Connections between Lie algebras and q-series

—areas which have felt his magic touch, and some that he has gloriously transformed. There is currently an explosion of activity in many of these areas and a significant portion of such work having its roots in the writings of Ramanujan. In particular, Ramanujan-type identities have arisen in studies of Lie algebras, and some have found applications in physics. Therefore I felt that it was an opportune time to start a journal devoted to the topics listed above.

In August 1993, at the joint Annual Meeting of The American and Canadian Mathematical Societies in Vancouver, Canada, I discussed this matter with Professors George Andrews and Bruce Berndt and they liked the idea. In the Autumn of 1993, based on a suggestion of Bruce Berndt, John Martindale, Editor, Science and Technology Division of Kluwer Academic Publishers, called me, and expressed interest in starting a new journal along these lines. I then had long telephone conversations with Professors George Andrews, Bruce Berndt, Basil Gordon, Jonathan Borwein and Peter Borwein about the prospects of a new journal devoted to Ramanujan type analysis. These mathematicians felt that it was a good idea to start such a journal. Bruce Berndt who has edited Ramanujan's Notebooks, and George Andrews who has plans to similarly edit the Lost Notebook in collaboration with Bruce Berndt, foresee several years of intense research activity in these areas and therefore felt that the new journal would be in the interests of high quality research. Basil Gordon, who is one of two Editors-in-Chief of the Journal of Combinatorial Theory, Series A, expressed the opinion that there is currently a worldwide explosion of ideas in these areas, and so if a new journal is to be started, this is the time to do so. Both Jonathan and Peter Borwein felt that the new journal would have a positive effect in bringing together many high quality research papers in these areas which would otherwise be scattered in the literature. These leading mathematicians who expressed positive views also said that they would support the journal by serving on its editorial board.

Encouraged by this support, I sent out a proposal in March 1994 to about 100 top mathematicians worldwide. Many responded with enthusiasm and so in March 1995, Dr. David Larner of Kluwer's head office at Dordrecht in the Netherlands and John Martindale of Kluwer's office in Norwell, Massachusetts, came to Florida for discussions and asked me to be the Editor-in-Chief of the newly proposed journal and form the editorial board. During the next two months I contacted the following mathematicians who all agreed to be on the editorial board:

George Andrews—The Pennsylvania State University
Richard Askey—University of Wisconsin
Bruce Berndt—University of Illinois
Frits Beukers—Rijksuniversiteit te Utrecht

Jonathan Borwein—Simon Fraser University
Peter Borwein—Simon Fraser University
David Bressoud—Macalester College
Peter Elliott—University of Colorado
Paul Erdös—Hungarian Academy of Sciences
Frank Garvan—University of Florida
George Gasper—Northwestern University
Dorian Goldfeld—Columbia University
Basil Gordon—University of California, Los Angeles
Andrew Granville—University of Georgia
Adolf Hildebrand—University of Illinois
Mourad Ismail—University of South Florida
Marvin Knopp—Temple University
James Lepowsky—Rutgers University
Lisa Lorentzen—The Norwegian Institute of Technology
Jean-Louis Nicolas—Université Claude Bernard
Alfred van der Poorten—Macquarie University
Robert Rankin—University of Glasgow
Gerald Tenenbaum—Université de Nancy
Michel Waldschmidt—Université P et M Curie
Don Zagier—Max Planck Institüt für Mathematik
Doron Zeilberger—Temple University.

In particular, Bruce Berndt and my colleague Frank Garvan, kindly agreed to act in the capacity of Coordinating Editors.

This is a very distinguished editorial board because its members are leaders in their areas of research. In particular, some have distinguished appointments at their respective universities and some have won prestigious prizes for their achievements. For instance, George Andrews is Evan Pugh Professor of Mathematics at Penn. State University while Richard Askey is Gabor Szëgo Professor in Wisconsin. A few years ago, Dorian Goldfeld and Don Zagier won the prestigious Cole Prize of The American Mathematical Society. Professor Paul Erdös (who unfortunately passed away since the editorial board was formed) was one of the legends of twentieth century mathematics. He won numerous awards and recognitions including the Cole Prize and Wolf Prize. And in the summer of 1996, Bruce Berndt was awarded the Steele Prize of The American Mathematical Society for the monumental task of editing Ramanujan's notebooks. With such experts on the editorial board, the journal will surely maintain a very high standard and attain its objectives. I shall describe the aims and scope of the journal now.

Aims and Scope The Ramanujan Journal will publish original research papers of the highest quality in all areas of mathematics influenced by Ramanujan with particular emphasis in *number theory, combinatory analysis, modular forms, and q-series*. It is a peer reviewed journal committed to timely publication. There will be one volume per year of about 400 pages consisting of four quarterly issues. Topics of interest include but are not restricted to those listed at the beginning. Thus the

journal will simultaneously have a wide scope as well as a sense of focus—wide scope because of the displayed topics, and focus because the journal is not devoted to all areas of mathematics. At present there are no journals devoted primarily to these topics and the Ramanujan Journal will fill this need. Ramanujan's work revealed unexpected connections between many of the listed areas. In a similar spirit, many papers which will appear in this journal, will reinforce such connections and establish links with new fields. As the sphere of Ramanujan's influence expands, the journal will suitable enlarge its scope to keep abreast on new developments.

At present there are two major journals devoted to Number Theory—The Journal of Number Theory and Acta Arithmetica.[1] Although both these journals are concerned with all of number theory, the former has a stronger tradition in the algebraic aspect and the latter the analytic side of the subject. There are also journals devoted to all areas of mathematics but with a strong tradition in Number Theory, such as Journal für die Reine und Angewandte Mathematik, Mathematika and Acta Mathematica Hungarica. The major journals in Combinatorics include The Journal of Combinatorial Theory Series A and B, The European Journal of Combinatorics, Combinatorica, and Algebraic Combinatorics.[1] But there is no journal which emphasises the theory of modular forms. There is also no journal with a particular emphasis on q-series although the SIAM Journal of Mathematical Analysis has a tradition of publishing papers on special functions and q-series. Thus the Ramanujan Journal will have a wider scope than the specialised journals mentioned above and so will not be in direct competition with them.

There are currently two journals published in India which bear Ramanujan's name in the title. One is The Hardy-Ramanujan Journal which was founded in the 1970s by Professor K. Ramachandra who is now retired from the Tata Institute. This journal publishes one issue per year and concentrates on the analytical and other classical aspects of number theory. The other is The Journal of the Ramanujan Mathematical Society which was formed around Ramanujan's centennial and is devoted to all areas of mathematics. The newly formed Ramanujan Journal would complement the interests of these journals especially in the international scene.

The Ramanujan Journal will publish review articles occasionally by invitation. These reviews will have the effect of increasing the readership of the journal by showing that the topics of interest to the journal are important and the frontiers are expanding. In fact, the opening article in the first issue of the journal in January 1997 is a review paper by Professor George Andrews based on the History of Mathematics Lecture he gave at the Joint Annual Meetings of the American Mathematical Society and the Mathematical Association of America in Minneapolis in August 1994. In addition, periodically, The Ramanujan Journal will bring out special issues either a collection of papers presented at a conference or an issue in honour of a great mathematician. For instance, Issue 4 of Volume 1 to appear in October 1997 will be the Proceedings of an International Number Theory Symposium that was conducted

[1] After this article was written, the Annals of Combinatorics was launched in 1997, and the International Journal of Number Theory in 2005.

at Anna University in January 1996. Also the first two issues of Volume 2 to appear in January and April 1998 will be in memory of Professor Erdös.

During the past few years, I have greatly enjoyed helping in setting up and promoting the journal. I could not have done this without the total support of David Larner and John Martindale of Kluwer and the encouragement of many mathematicians, especially those on the editorial board. I also got immense pleasure in helping to design the cover of the journal which will have Ramanujan's handwriting in faded form in the background. This is taken from Ramanujan's last letter to Hardy wherein he mentions the discovery of the mock theta functions. Narosa Publishing House of New Delhi which brought out The Lost Notebook and other unpublished papers of Ramanujan during the centennial celebrations in 1987, very kindly granted permission to reproduce this.

The very mention of Ramanujan's name reminds us of the thrill of mathematical discovery. We have set ourselves a very high standard by having Ramanujan's name in the title of the journal. I hope that the mathematics which will fill the pages of this journal will be a source of excitement to experts and non-experts alike.

Addendum Dated September 2012 The *Ramanujan Journal*, which was launched by Kluwer Academic Publishers in 1997, has been published by Springer since 2005 after the merger of Kluwer and Springer. The journal which brought out one volume per year of four issues with about 100 pages per issue, now publishes three volumes per year of three issues per volume, with about 150 pages per issue. Thus the journal has tripled in size since the take over by Springer and my thanks to Ann Kostant and Elizabeth Loew of Springer for supporting this expansion. Also, some eminent mathematicians have joined the Editorial Board in view of this expansion. This rapid growth is a testimony to the leading position the journal occupies in its areas of coverage.

Krishnaswami Alladi
Editor-in-Chief

Chapter 25
A Pilgrimage to Ramanujan's Hometown

The district of Tanjore (*Tanjavur* in Tamil) in the state of Tamil Nadu in South India has been a seat of culture for several centuries. Tanjore has produced some of the greatest composers of *Carnatic* music, the classical music of South India. Tanjore is also very well known for art in various forms; in particular, Tanjore paintings of Hindu gods, which have crystal glass pieces imbedded in them, are much appreciated both for their special beauty and art value. The Tanjore district also has the greatest concentration of Hindu temples, and some of them are architectural marvels. It is in the midst of this region steeped in culture that the Indian mathematical genius Srinivasa Ramanujan lived in the town of *Kumbakonam* in Tanjore district and made some of his path-breaking discoveries. In December 2003, my wife Mathura and I had an opportunity to visit Kumbakonam, see Ramanujan's humble home from which a thousand theorems emerged, and visit several temples in that area including the one next to Ramanujan's home where he prayed regularly. In this article I shall describe our memorable visit to Kumbakonam and Tanjore. But to prepare a suitable background, I first will briefly describe Ramanujan's fascinating life story.

Ramanujan Srinivasa Ramanujan, one of the greatest mathematicians in history, belonged to an orthodox Hindu brahmin family in the town of Kumbakonam. He was born in Erode in South India on 22 December 1887. Erode was where his mother's family lived. It is the Hindu custom that even though after marriage a woman would live in her husband's home, she will go to back to her family home to give birth. Among the brahmins, there are two main subcastes in South India, the *Iyers* who worship Lord Shiva the Destroyer as the primary diety, and the *Iyengars*, who worship Lord Vishnu the Protector as the main diety. Ramanujan was an Iyengar, and from his parents he learnt many verses of the *Vedas*, the Hindu holy scriptures, as well as stories from Hindu folklore and epics. Ramanujan and

An abridged version of this article appeared in *FOCUS*, the newsletter of the Mathematical Association of America, in May 2005.

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Fig. 25.1 Krishnaswami Alladi and George Andrews in front of Srinivasa Ramanujan's home, Sarangapani Sannidhi Street, Kumbakonam; December 20, 2003

his family offered prayers regularly at the *Sarangapani temple* for Lord Vishnu in Kumbakonam, which was just down the street from his home (see Fig. 25.3).

Ramanujan showed his unusual talent for mathematics very early. Often in the middle of the night he would get up and write down mathematical formulae on a piece of slate lest he should forget them in the morning when he woke up. He would then record these marvelous formulae in his now famous notebooks. Ramanujan had a special veneration for the Goddess *Namagiri* of the temple in the neighbouring town of *Namakkal*, and we are told that the Goddess of Namakkal would come in his dreams and give him these formulae! In the town high school in Kumbakonam, Ramanujan's teachers realized that he was unusually talented, but they obviously could not understand or judge the importance of his discoveries. Ramanujan later moved to *Madras* (now called Chennai), the capital and largest city in Tamil Nadu, where he attended College. Although he was successful in high school, his obsessive preoccupation with mathematics led to a neglect of other subjects, and so he had to drop out of college. The advantage of being in Madras was that he could come in contact with persons, both Englishmen and Indians, who could appreciate his work. Some of them suggested that he should communicate his findings to leading mathematicians in England—India was a British colony at that time. The rest is history!

The two letters Ramanujan wrote to G.H. Hardy of Cambridge University are considered to be among the most significant in mathematical history. In these letters Ramanujan communicated hundreds of bewildering mathematical formulae he had discovered. Hardy and his peers in Cambridge were convinced by the letters that

Fig. 25.2 The only cot at Ramanujan's modest home in Kumbakonam. Ramanujan used to sit on this cot or on the window sill watching the street as he worked on his "sums" when he was a young boy; December 20, 2003

Ramanujan was a genius of the class of Euler or Jacobi. Hardy invited Ramanujan to Cambridge to work with him so that the untutored genius could be given a proper sense of direction. Orthodox Hindus believed that it was a sin to cross the oceans, and so Ramanujan declined this invitation; his mother would not give him permission to go. But Hardy persisted. One night, his mother had a dream in which she saw Ramanujan being honoured by foreigners in a great assembly. In that same dream the Goddess of Namakkal ordered the mother not to stand in the way of her son's recognition! Thus with his mother's permission, Ramanujan sailed for England in 1914.

Hardy, being an agnostic, dismissed the Goddess of Namakkal stories as mere fables. However, I should point out that it is very natural for Hindus to accept such legends. Hindu belief is that there is a divine origin to every aspect of knowledge, including music, and that is why much of Hindu classical music is devotional. Hindus believe in the story that Kalidasa, the greatest sanskrit poet, was transformed from an uneducated cowherd to a poet par-excellence overnight because the Goddess Kali wrote her blessings on his tongue. Thus it is very natural for Hindus to accept the Goddess of Namakkal as a divine origin of Ramanujan's great discoveries.

In England, within a short span of five years, Ramanujan wrote several fundamental papers, some with Hardy, that revolutionized various areas of mathematics. But conditions were difficult in England at that time, worsened by the First World War. Ramanujan was a strict vegetarian, and food to suit his dietary needs was difficult to get in wartime England. He also did not take care to protect himself from the

Fig. 25.3 The majestic *gopuram of* the Sarangapani Temple as seen from the front of Ramanujan's home on Sarangapani Sannidhi Street; December 20, 2003

cold English winters. Thus he had to return to India in 1919, a very sick man. Hardy was concerned that Ramanujan might not live long, and so he worked hard to get him elected as Fellow of the Royal Society (FRS) in 1918. What a recognition for someone who did not even have a college degree! Shortly after Ramanujan returned to India, he died in April 1920 in Madras. In his 32 years, Ramanujan had made outstanding contributions and was recognized with the highest honour that any academician in the British Commonwealth could aspire for, the election as Fellow of the Royal Society.

SASTRA University and Ramanujan's Home The Shanmugha Arts, Science, Technology and Research Academy (SASTRA), is a private (deemed) university whose main campus is located in the town of Tanjore, after which the district is named. SASTRA University was created about 15 years ago as a private educational institution. Unlike public colleges and universities in India, where most admission is based on a quota system for certain underdeveloped segments of society, admission to private educational institutions like SASTRA is based on merit. Thus in a short

Fig. 25.4 l to r: Krishnaswami Alladi, Prof. George Andrews (Penn. State University) and SASTRA Vice-Chancellor R. Sethuraman in front of a bust of Srinivasa Ramanujan at SASTRA University, Kumbakonam; December 22, 2003

span of time, SASTRA attracted some of the brightest students and the best teachers, and therefore grew both in size and quality to attain the status of a deemed university. Owing to this successful growth, SASTRA recently opened a second campus at Ramanujan's hometown, Kumbakonam.

In 2003 SASTRA University purchased the home of Ramanujan and will maintain it as a museum. Ramanujan is an idol and inspiration to all students in India, and hence the preservation of his home was essential to the spirit and hope of Indian intellectuals. SASTRA also created a Srinivasa Ramanujan Centre which has a library that contains several Ramanujan memorabilia, as well as books, papers, and journals relating to his work. Since a university has now purchased Ramanujan's home, we now have the active involvement of administrators, academicians, and students, in the preservation of Ramanujan's legacy for posterity.

To mark the occasion of the purchase of Ramanujan's home, the Srinivasa Ramanujan Centre of SASTRA University conducted an International Conference on Number Theory and Secure Communications during 19–22 December 2003, where I was invited to lecture on my work (see Fig. 25.4). Other plenary speakers at this conference included George Andrews from The Pennsylvania State University, Noam Elkies of Harvard University, Samuel Wagstaff of Purdue University and Antol Balog of The Hungarian Academy of Sciences. George Andrews gave the Opening Lecture of the conference as well as the concluding Ramanujan Commemmoration Lecture on 22 December, Ramanujan's birthday. The conference was supported by the Indo-US Forum, the Number Theory Foundation, and several funding agen-

Fig. 25.5 l to r: Professor George Andrews (Penn. State University), Mrs. Mathura Alladi, and Professor Noam Elkies (Harvard University) at Sterling Resorts outside Kumbakonam; December 22, 2003

cies in India. The President of India, Dr. Abdul Kalam, inaugurated the conference and declared open Ramanujan's home as a museum and national treasure. My wife Mathura, who is a Bharathanatyam dancer and teacher, and I were happy to have an opportunity to visit this home and the wonderful temples of the Tanjore district.

Accommodation and Sightseeing All the delegates with their families who came from overseas were accommodated at Sterling Resorts in *Swami Malai* near Kumbakonam. Swami Malai derives its name from a temple there for Lord Subramanya, who as a child explained the deep meaning of the sacred Hindu syllable *Om* to his father Lord Shiva. The significance of this story is that knowledge has no age barriers. *Swami*, which means God, refers here to Lord Subramanya, and *Malai*, which means mountain in Tamil, refers to a little hill on top of which the temple is located.

Sterling Resorts is an amazing place. It is a set of cottages with tiled roofs, in a farm or plantation setting (see Fig. 25.5). There are plenty of banana trees all around and several cows on the premises of the resort. Thus milk and all milk products required for guests are obtained fresh from the resort itself. Also, delicious Indian food is served Indian style on banana leaves. On the resort grounds there is a magnificent white statue of Lord Shiva with his young son Lord Subramanya on his lap whispering the meaning of *Om* into his father's ear. When all guests arrived at the resort, we were given a very warm traditional welcome, namely, we were not only greeted with garlands and flowers, but trained staff at the resort in traditional Indian dress, washed and massaged our feet with fragrant water. I suppose the tradition came

about because in the past travellers used to arrive by foot, and so this foot massage was a welcome relief. We arrived by a van from Madras, yet we immensely enjoyed and appreciated this traditional welcome!

Although the Sterling Resorts has an old fashioned setting, the rooms have all amenities, including a TV, telephone, hot water heater, etc. There are no note pads near the telephone to take down messages. Instead, you are provided a slate and a piece of chalk. When I went to bed I hoped that like Ramanujan, I too would get a formula in my dream so that in the middle of the night I could get up and write it down on the slate with the piece of chalk!

Seeing Ramanujan's home was itself a dream come true. What an inspiration to see this small humble home from where so many significant mathematical discoveries poured forth. The home has only three rooms surrounding a small courtyard: a bedroom, a kitchen, and a dining area. In the back there is a well and a bathroom. This was a typical village home where as was customary, a family of four to eight lived here. Not every one could sleep in the bedroom, and so members of Ramanujan's family slept in the courtyard as well. The home is located on Sarangapani Sannidhi Street, so named, because Sarangapani Temple is located at the end of the street. There is a window in the bedroom overlooking the street. As a boy, Ramanujan used to sit on the bedroom cot doing mathematics and watching the passers by on the street through this window (see Fig. 25.2). Mathura and I had the pleasure of seeing Ramanujan's home along with Professor George Andrews of The Pennsylvania State University, one of the world's greatest authorities on Ramanujan's work (see Fig. 25.1).

After seeing Ramanujan's home, we went to the Sarangapani Temple to offer *Archana* to the Lord. Archana is a special form of prayer in which the priest mentions the family background (*sankalpa*) of the devotee on whose behalf he is offering the prayer. It was a very pleasant surprise that the high priest not only asked Professor Andrews to join us in the sanctum sanctorum of the temple but included his name in the sankalpa as well. In many Hindu temples of South India, even today, only Hindus are allowed into the inner sanctum. This recent practice of allowing others into the sanctum was a very pleasant development we saw in the Tanjore district at all temples we visited.

Tanjore town, which is about half an hour from Kumbakonam by car, has one of the most impressive temples in all of South India, the *Brihadeeswara temple* for Lord Shiva. When you enter its grounds, you see a massive *Nandi*, or bull, which is the guardian diety in all Shiva temples. As we entered the Brihadeeswara temple, we saw an *abhisheka* (daily purification bath) being performed for the Nandi. When an abhisheka is performed, the diety is washed with milk, yoghurt, honey, and in this case also *veeboothi* (holy ash of Lord Shiva), and finally with water. All through the abhisheka, the appropriate verses from the *vedas* (Hindu holy scriptures) are chanted. Imagine the amount of milk, yoghurt and honey used in the abhisheka of a massive Nandi!

After offering prayers at the Brihadeeswara temple, our hosts at SASTRA took us to a few shops in Tanjore town containing a wonderful collection of local art. Mathura wanted to purchase a Tanjore painting during a visit there. So we actually

went to the home of a local artisan and bought a lovely painting of Lord Krishna. This now adorns the wall of the *puja* (worship) room in our home in Gainesville, to remind us every day of the rich cultural experience we had in Tanjore district. The visit was also truly memorable because it provided us an opportunity to see the home of Ramanujan, thereby making it an unforgettable mathematical pilgrimage.

Chapter 26
The First SASTRA Ramanujan Prizes

It was one of the most thrilling moments of my life to be in Kumbakonam, the hometown of the Indian mathematical genius Srinivasa Ramanujan, and participate in a function when the First SASTRA Ramanujan Prizes were awarded to two of the most brilliant young mathematicians for outstanding contributions to areas of mathematics influenced by Ramanujan. In the preface to the first issue of the Ramanujan Journal I said “the very mention of Ramanujan’s name reminds us of the thrill of mathematical discovery.” Ramanujan is an inspiration for mathematical aspirants and researchers the world over and a role model and idol for all in India where he is a household name. There can be no better way to commemorate Ramanujan than to award these prizes for exceptional mathematical creativity at a very young age. SASTRA University under the leadership of Vice Chancellor Prof. R. Sethuraman has made laudable efforts in fostering the legacy of Ramanujan and supporting mathematical research by first purchasing Ramanujan’s home in 2003 and maintaining it as a museum, by conducting annual international conferences in areas of mathematics influenced by Ramanujan, and by launching the Ramanujan Commemoration Lectures, which are talks of wide appeal delivered by very eminent mathematicians annually on 22 December, Ramanujan’s birthday. Their latest step is the creation of the SASTRA Ramanujan Prize, which is one of the finest ways to recognise path-breaking contributions to mathematics, and SASTRA has to be congratulated for this.

The age limit for the prize was set at thirty two in order to recognise doctoral and post-doctoral research, and also because Ramanujan achieved so much in his brief life of thirty two years. This age limit might appear to be too severe a restriction,

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This article is a slightly expanded version of a talk given at the award ceremony at SASTRA University, Kumbakonam, on December 20, 2005. An abridged version of this article appeared in FOCUS, the newsletter of the Mathematical Association of America, in May 2006. On 20 December 2005, the First SASTRA Ramanujan Prizes were awarded at Kumbakonam, India, Ramanujan’s home town, to Professors Manjul Bhargava (Princeton) and Kannan Soundararajan (Michigan). This article describes the goals of the prize, the events leading up to and including the prize ceremony, and the accomplishments of the winners.

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but it is not so, because in mathematics, more so than in other fields, path-breaking work is often done by very young researchers. If one looks at the lives of outstanding mathematicians, one notices in a vast majority of cases that their very best work was done in their youth. This is not to say that they did not continue to do influential work in later years. Mathematics is as much an art as it is a science, and it is a subject in which one explores the structures, the symmetries and the inter relationships for their intrinsic beauty. Youthful minds are capable of taking fantastic leaps of imagination.

The decision to create the prize was made during a discussion I had with the Vice-Chancellor during the International Conference on Fourier Analysis and Number Theory at SASTRA University, Kumbakonam, in December 2004, which I had the pleasure of inaugurating. The Vice-Chancellor announced that this annual prize of $10,000 will be first awarded at the International Conference on Number Theory and Mathematical Physics at SASTRA's Srinivasa Ramanujan Centre in Kumbakonam in December 2005. I was invited by SASTRA to form and head the 2005 Prize Committee.

In forming the committee the desire was to assemble a group of very eminent and experienced mathematicians from different countries to reflect a truly international character, and whose research expertise would broadly span several areas of mathematics influenced by Ramanujan. The 2005 SASTRA Ramanujan Prize Committee consisted of Krishnaswami Alladi, Chair (University of Florida), Mahindra Agarwal (IIT, Kanpur), George Andrews (Pennsylvania State University), Jean-Marc Deshouillers (University of Bordeaux), Tom Koornwinder (University of Amsterdam), James Lepowsky (Rutgers University) and Don Zagier (Max Planck Institute, Bonn, and College de France).

The Committee was pleased to receive several excellent nominations of brilliant young mathematicians from around the world supported by leaders in the field. It turned out that two candidates of Indian origin emerged as the best in this international competition, Manjul Bhargava of Princeton University and Kannan Soundararajan of the University of Michigan. The decision was to award prizes to both Bhargava and Soundararajan whose areas of research are algebraic number theory and analytic number theory, respectively. Thus the prizes recognised seminal research in two of the main branches of number theory.

The SASTRA Ramanujan Prizes of $10,000 each were awarded to Bhargava and Soundararajan on 20 December, 2005 during the inauguration of the International Conference on Number Theory and Mathematical Physics at Kumbakonam by Dr. Aurobindo Mitra, Executive Director of the Indo-US Forum for Science and Technology, which provided significant support for the conference (see Figs. 26.1, 26.2, and 26.3). In handing out the prizes, Dr. Mitra described many new research programs supported by the Indo-US Forum.

The opening lecture of the conference was a talk by Soundararajan on "Large character sums: The Polya–Vinogradov theorem." The conference concluded with the Ramanujan Commemoration Lecture by Bhargava in which he announced his most recent spectacular result (joint with Jonathan Hanke), namely, the complete determination of all universal quadratic forms, thereby solving a problem which

Fig. 26.1 Krishnaswami Alladi, Chair SASTRA Ramanujan Prize Committee, speaking at the First SASTRA Ramanujan Award Ceremony; December 20, 2005

has its origins in Ramanujan's work. It was fitting that Bhargava announced this on Ramanujan's birthday (22 December) in Ramanujan's hometown! I interviewed Bhargava and wrote a report of his lecture which appeared in The Hindu, India's National Newspaper, the next day:

http://www.hindu.com/2005/12/23/stories/2005122306251400.htm.

I will now describe briefly the career highlights and research accomplishments of Bhargava and Soundararajan.

Manjul Bhargava was an undergraduate at Harvard University from where he graduated with highest honours (summa cum laude) in mathematics in 1996. He was Harvard University Salutatorian and won the Hoopes Prize for excellence in scholarly work and research at Harvard in 1996. He was also awarded the Frank and Brennie Morgan Prize of the American Mathematical Society for undergraduate research in 1996. He then went on to do his Ph.D. at Princeton University under the direction of Prof. Andrew Wiles (of Fermat's Last Theorem fame). Bhargava wrote a phenomenal Ph.D. thesis in Princeton in 2001, in which he described his discovery of higher-order composition laws. His thesis was published as a series of four papers in Annals of Mathematics, one of the most exclusive mathematics journals.

Gauss, the Prince of Mathematicians, had constructed a composition law from binary quadratic forms. Bhargava introduced entirely new and unexpected ideas that led to his discovery of such composition laws for forms of higher degree. Bhargava then applied these composition laws to solve a new case of one of the fundamental questions of number theory, that of the asymptotic enumeration of number fields

Fig. 26.2 Professor Manjul Bhargava (Princeton University), receiving the First SASTRA Ramanujan Prize. From l to r: SASTRA Vice-Chancellor R. Sethuraman, DST Chairman Dr. Aurobinda Mitra, Prize Winner Manjul Bhargava, Prize Committee Chair Krishnaswami Alladi, and SASTRA Dean P.V. Ramakrishnan; December 20, 2005

of a given degree. The question is the trivial for $d = 1$, and Gauss himself solved the case $d = 2$ in 1801. Then in 1971 Davenport and Heilbronn solved the case for $d = 3$. Bhargava has solved the $d = 4$ and $d = 5$ cases, which previously had resisted all attempts. Bhargava also applied his work to make significant progress on the problem of finding the average size of ideal class groups and the related conjectures of Cohen and Lenstra. Bhargava's research has created a whole new area of research in a classical topic that has seen a very little activity since the time of Gauss.

Naturally, Bhargava received several recognitions for work of such great significance. After his Ph.D., he was appointed Long Term Prize Fellow at the CLAY Mathematics Institute. It is the CLAY Institute which has created the Millennium Prizes of $1 million each for seven of the most outstanding problems in mathematics. For his revolutionary Ph.D. work, Bhargava was awarded the Blumenthal Prize of the American Mathematical Society in January 2005. In early December 2005 he received the Clay Prize at a ceremony at Oxford University. Bhargava was appointed Full Professor at Princeton University at the age of 28 and is the youngest to hold that high rank in that prestigious institution.

Kannan Soundararajan's first publications were three papers that appeared in 1992 based on work that he did a few years prior to that while he was a student at Padma Seshadri High School in Nungambakkam in Madras. In one of those papers that appeared in the Journal of Number Theory, he improved an inequality on

Fig. 26.3 Professor Kannan Soundararajan (University of Michigan), receiving the First SASTRA Ramanujan Prize. From l to r: SASTRA Vice-Chancelor R. Sethuraman, DST Chairman Dr. Aurobinda Mitra, Prize Winner Kannan Soundararajan, Prize Committee Chair Krishnaswami Alladi, SASTRA Dean P.V. Ramakrishnan; December 20, 2005

multiplicative functions that I had proved in collaboration with Paul Erdös, one of the legends of twentieth century mathematics, and Jeff Vaaler.

Soundararajan joined the University of Michigan, Ann Arbor, in 1991 for undergraduate studies, and graduated with highest honours in 1995. He has made brilliant contributions to several areas of analytic number theory that include multiplicative number theory, the Riemann zeta function and Dirichlet L-functions, and more recently with the analytic theory of automorphic forms and the Katz–Sarnak theory of symmetric groups associated with automorphic forms. As an undergraduate at the University of Michigan, Soundararajan made two significant contributions. First in joint work with R. Balasubramanian, he proved a famous conjecture of Ron Graham in combinatorial number theory. Next he obtained fundamental results on the distribution of zeros of the Riemann zeta function. For his undergraduate research he was awarded the Morgan Prize of the American Mathematical Society in 1995, the very first year this prize was instituted.

Soundararajan then joined Princeton University in 1995 to do his Ph.D. under the guidance of Professor Peter Sarnak, one of the foremost number theorists in the world today. In his Ph.D. thesis, Soundararajan proved the spectacular result that more than seven-eighths of the quadratic L-functions have zeros at the critical point $s = 1/2$. A part of his Ph.D. thesis is published in the Annals of Mathematics. More recently, in a paper with Brian Conrey in Inventiones Mathematicae, Soundararajan

proved that a positive proportion of Dirichlet L-functions have no zeros on the real axis within the critical strip. In another paper with Ken Ono, also in Inventiones, he proved assuming the generalized Riemann hypothesis, a certain conjecture of Ramanujan on ternary quadratic forms. Soundararajan is also a leading expert on random matrix theory. His recent work with Hugh Montgomery shows that prime numbers are distributed normally but with a variance that is surprisingly different from classical heuristics.

Soundararajan has received numerous awards and recognitions starting with the Silver Medal in the 1991 International Mathematics Olympiad. As a graduate student at Princeton, he held a prestigious Sloan Foundation Fellowship. For his outstanding contributions to analytic number theory, he was awarded the Salem Prize in 2003. Shortly after his Ph.D. he was awarded a Five Year American Institute of Mathematics (AIM) Fellowship, the very first year the fellowship was launched. Soundararajan is currently a Full Professor at the Mathematics Department, University of Michigan, Ann Arbor.

Bhargava and Soundararajan have several things in common. Both received the Morgan Prize of the American Mathematical Society for undergraduate research. Both received their Ph.D.s from Princeton University. And now, both have received the SASTRA Ramanujan Prizes. By awarding this prize to these two brilliant young mathematicians, an exceptionally high standard has been set in the true spirit of Ramanujan.

Addendum, September 2012 Since its launch in 2005, the SASTRA Ramanujan Prize has been given every year around December 22 (Ramanujan's birthday) at Kumbakonam (Ramanujan's hometown) at an international conference at SASTRA University. In view of the fact that 2012 is the 125-th birth anniversary of Srinivasa Ramanujan, this year alone the prize is being given outside of Kumbakonam in New Delhi (India's capital) on December 22 at an international conference on the legacy of Ramanujan. The winners since 2005 are:

2005: Manjul Bhargava (Princeton) and Kannan Soundararajan (Michigan)
2006: Terence Tao (University of California, Los Angeles)
2007: Ben Green (Cambridge University)
2008: Akshay Venkatesh (Stanford University)
2009: Kathrin Bringmann (University of Cologne)
2010: Wei Zhang (Harvard University)
2011: Roman Holowinsky (Ohio State University)
2012: Zhiwei Yun (Massachusetts Institute of Technology and Stanford University)

The reputation of any prize is determined by the caliber of the winners. Each of the winners is a world leader in his or her field of study, and so this prize has become one of the most prestigious and coveted awards. I have had the privilege of being the (non-voting) Chair of the Prize Committee since 2005. I am indebted to several leading mathematicians for supporting the prize either by sending in outstanding nominations or by serving on the prize committee.

Chapter 27
Ramanujan's Growing Influence

Srinivasa Ramanujan's influence seems only to increase and not diminish with time. In the early part of the twentieth century, Ramanujan was perceived as a mathematical phenomenon emerging from the East by the great stalwarts like G.H. Hardy and others at Cambridge University. Although they admired Ramanujan for his genius, Hardy and his contemporaries could not measure the full significance of Ramanujan's discoveries and the eventual impact this would have. Over the years the magnitude and importance of Ramanujan's mathematics has been realised, and its impact in various branches of mathematics, such as Number Theory, Combinatorics, Analysis, Modular Forms, Basic Hypergeometric Series, and Special Functions, is deep and everlasting. Indeed, Ramanujan's identities have made their presence in other subjects like physics and computer science.

Hardy nurtured Ramanujan and lectured often on Ramanujan's work. Hardy's Twelve Lectures on Ramanujan's is a model of fine mathematical exposition. These lectures, along with Ramanujan's Collected Papers, served as the principal source of inspiration and reference for many years for those who desired to understand the remarkable work of the Indian genius. In the last few decades there has been several significant publications expanding on Ramanujan's work and therefore has impacted a much wider community of research mathematicians. We owe a special debt of gratitude to the great Trinity of the World of Ramanujan: (i) to George Andrews for explaining the significance of many of Ramanujan's identities especially in the context of partitions and for discovering Ramanujan's "Lost Notebook" and helping us understand hundreds of deep identities contained therein including those on mock theta functions, (ii) to Bruce Berndt for editing Ramanujan's Notebooks in Five Volumes, and (iii) to Richard Askey for providing the broad picture of how Ramanujan's work fits in the world of Special Functions. Thus the present day researcher can easily enter the mansion of Ramanujan's theorems and make connections with current research.

This article appeared in *The Hindu*, India's national newspaper in December 2003 in connection with the start of the annual Ramanujan conferences at SASTRA University.

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The Ramanujan Centennial, celebrated in 1987, was an occasion when mathematicians around the world gathered to pay homage to the Indian genius. The centennial celebrations showed clearly how alive Ramanujan is in current mathematical research and how much an inspiration he was to celebrated mathematicians like Atle Selberg. While attending the centennial, I was inspired to create something which would simultaneously be a tribute to Ramanujan and would connect Ramanujan to current research developments continuously. Thus I got the idea to launch *The Ramanujan Journal, an international journal dedicated to all areas of mathematics influenced by Ramanujan*. This desire of mine became a reality in 1997 after this idea received support from the international community of experts, some of whom serve on the Editorial Board with me.

In the last decade, Ramanujan has made an impact beyond mathematics on society in general. Of course, throughout India, Ramanujan's remarkable story is well known, and Ramanujan is a hero to every eager Indian student of mathematics. But with the publication of Robert Kanigel's book, *The man who knew infinity*, Ramanujan's story reached out to society around the world, and the importance of this impact cannot be underestimated. Subsequently, Bruce Berndt and Robert Rankin have published two wonderful books. The first one called *Ramanujan—Letters and Commentary* collects various letters written to, from, and about Ramanujan, and makes details commentaries on the letters. For instance, if a letter contains a mathematical statement, there is an explanation of the mathematics with appropriate references. If there is a statement about Ramanujan being elected Fellow of the Royal Society (FRS), then there is a discussion about the procedures and practices for such an election. The second book called *Ramanujan—Essays and Surveys* is a collection of excellent articles by various experts on Ramanujan's life and work. The book also contains articles about certain individuals who played a major role in Ramanujan's life. Thus both books will appeal not only to mathematicians, but to students and lay persons as well.[1]

In what other ways will we see Ramanujan influence us in the future? Courses on Ramanujan's work are regularly offered at various universities where there groups of experts work on Ramanujan's manuscripts. In writing the Editorial for the first issue of The Ramanujan journal I said, "The very mention of Ramanujan's name reminds us of the thrill of mathematical discovery." Now with the appearance of these books that are now having an impact on society in general, it may not be an exaggeration to predict that in the future, *Ramanujan* will be a topic or subject that undergraduate mathematics students worldwide may be studying regularly!

The latest big event in the world of Ramanujan is the recent acquisition of Ramanujan's home in Kumbakonam by SASTRA University. This private university that was founded recently has grown by leaps and bounds. We owe special thanks to Professor R. Sethuraman, Vice-Chancellor of SASTRA University, and his family for taking steps to ensure that Ramanujan's home will be properly maintained. Since

[1] A comprehensive CD on Ramanujan edited by Srinivasa Rao was produced in 2004 containing among other things scanned copies of Ramanujan's notebooks and papers, the five volume work of Berndt, and hundreds of letters and documents related to Ramanujan.

a university has purchased Ramanujan's home, we now have the active involvement by administrators, academicians, and students, in the preservation of Ramanujan's legacy for posterity. It is indeed fitting that to mark this occasion, an International Conference on Number Theory and Secure Communications is organised in Kumbakonam during 20–22 December 2003, by SASTRA University, and that the inauguration of this conference is by the President of India, Dr. Abdul Kalam. The opening lecture of the conference on 20 December is by Professor George Andrews, as is the concluding Ramanujan Lecture on the morning of 22 December, Ramanujan's 116th birth anniversary. When the Ramanujan Centennial was celebrated in 1987 in Madras, India's Prime Minister Rajiv Gandhi inaugurated the conference, released the printed version of Ramanujan's Lost Notebook, and presented the first copy to George Andrews. And now we have the President of India Dr. Kalam inaugurating this conference and declaring Ramanujan's home as a museum.

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